On the local time density of the reflecting Brownian bridge

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Abstract

Expressions for the multi-dimensional densities of Brownian bridge local time are derived by two different methods: A direct method based on Kac's formula for Brownian functionals and an indirect one based on a limit theorem for strata of random mappings.

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1 Introduction

Throughout this paper, the standard Brownian motion (BM) will be denoted by $x(t)$. The reflecting Brownian bridge (rBB) is the process $x^+(t)$ which is identical in law to $(|x(t)-tx(1)|, 0 \le t \le 1)$.

We are interested in the process $(\tau^+(a), a \ge 0)$ where $\tau^+(a)$ is the total local time of $x^+(t)$ at level a, defined by

$$\tau^+(a) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_0^1 I_{[a,a+\varepsilon]}(x^+(t))\, dt.$$

Several representations of the one dimensional density of this process are known. Though there is no direct study of this process, results on symmetric random walks or random mappings ([2, 7]) yield various density representations (see [21, 22, 7]).

Apart from the random mapping (see Sec. 2.3), applications of the rBB local time can be found in the analysis of Shellsort : (see Louchard [17, Sec. 2]). We need the distribution of the number $I(2n)$ of inversions in a 2-ordered permutation of the $2n$ values $\{1 \cdot\cdot 2n\}$ (i.e. permutation consisting of two interleaved sorted permutations). Position i of the odd part of the permutation contains value k if the path U_n corresponding to the permutation satisfies $U_n(i) = k - i$ (this is Knuth's correspondence between 2-ordering and path in a lattice, see Knuth [15, p. 87]). Now $U_n([nt] - nt)/\sqrt{2n} \Rightarrow BB(t)$ and $I(2n) = \sum_{i=1}^n |i - U_n(i) - 1|$.

The local time of the rBB at $\ell/\sqrt{2n}$ corresponds to the number of positions i containing $2i + \ell$ or $2i - \ell$.

The rBB local time corresponds also to the number of jumps at some level of the empirical distribution, in the context of the classical Kolmogorov-Smirnov distribution function.

Denote by $f_x(y)$ the density of $\tau^+(x)$. Then in [7] we find the representation

$$f_x(y) = \frac{1}{i\sqrt{2\pi}} \int_S \frac{e^\alpha}{2\cosh^2(x\sqrt{2\alpha}\,)} \exp\left(-\frac{y}{\sqrt{2}} \frac{\sqrt{\alpha} e^{x\sqrt{2\alpha}}}{\cosh(x\sqrt{2\alpha}\,)} \right) d\alpha \tag{1}$$

where $S := (a - i\infty, a + i\infty)$, $a > 0$, is a straight line parallel to the imaginary axis, which is the Brownian bridge analogon to the density presented in [5, 13] for the Brownian excursion. We will

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generalize this formula to several dimensions and offer two approaches: the first one is a direct computation by means of Kac's formula for Brownian functionals and the second one is based on the fact that the process consisting of the – suitably normalized – strata of a random mapping converges weakly to Brownian bridge local time.

The paper is organized as follows. In Sec. 2 we summarize basic notations and methods. Some preliminary formulas based on Kac's formula and their inversion are given in Sec. 3. Sec. 4 is devoted to the general multi-dimensional density. The Brownian excursion analogon of this problem has been treated in [12]. Thus we will keep our presentation rather brief and refer to [12] for details.

We would like to mention that MAPLE was of great help in computing some complicated expressions (with some guidance of course).

2 Basic notations and known results

2.1 Kac's formula for Brownian functionals

Denote by $\eta(t)$ any of the processes defined in the previous section. Then we will use the notation

$$E_a[B(\eta)] := \Pr[B|\eta(0) = a]$$

where $B(\eta)$ is an event belonging to the Borel field generated by $\eta(t)$. Furthermore denote by $\mathcal{L}_\alpha(f(x)) := \int_0^\infty e^{-\alpha x} f(x)\,dx$ the Laplace transform of $f(x)$. Then the classical density (for $\eta(t) = x(t)$)

$$p(t,x,y)dy := E_x[x(t) \in dy] = \frac{1}{\sqrt{2\pi t}} \exp\left[-\frac{(x-y)^2}{2t}\right] dy$$

implies

$$\mathcal{L}_\alpha(p(t,x,y)) = \frac{\exp(-\sqrt{2\alpha}|x-y|)}{\sqrt{2\alpha}} \tag{2}$$

where the Laplace transform is taken with respect to t.

Let $h \geq 0$ be a piecewise continuous function and let $\mathcal{G}$ be the differential operator

$$(\mathcal{G}u)(a) := \frac{1}{2}u''(a) - h(a)u(a).$$

Kac's formula states that, for $\alpha > 0$ and $f \in C(R^1)$.

$$u(a) = E_a \int_0^\infty e^{-\alpha t} \exp\left(-\int_0^t h[x(s)]\,ds\right) f(x(t))\,dt \tag{3}$$

is the bounded solution of

$$(\alpha - \mathcal{G})u = f. \tag{4}$$

The solution of (4) is given by $u(a) = \int G(a,b)f(b)\,db$ where the Green function G is given by

$$G(a,b) = G(b,a) = 2W^{-1}g_1(a)g_2(b), \qquad a \leq b,$$

where $0 < g_1 \in\uparrow, 0 < g_2 \in\downarrow$ are independent solutions of $\mathcal{G}g = \alpha g$ and W is their constant positive Wronskian: $W = g_1'g_2 - g_1g_2'$. (See Itô and McKean [14, par 2.6], Louchard [18])

If we add to h in (3) another function h^*, the modified Green function $G^*(a,b)$ satisfies the relation

$$G^*(a,b) = G(a,b) - \int G(a,x)h^*(x)G^*(x,b)\,dx$$

(see Itô and McKean [14, p. 67]).

Particularly, for $h^*(x) = \gamma I_{(\xi,\eta)}(x)/(\eta-\xi)$ (where $I_{(\xi,\eta)}$ is the indicator function of the interval (ξ, η) with $\eta > \xi$) we obtain by letting $\eta \downarrow \xi$

$$G^*(a,b) = G(a,b) - \gamma \frac{G(a,\xi)G(\xi,b)}{1+\gamma G(\xi,\xi)}. \tag{5}$$

This corresponds to adding $\gamma t^+(t,\xi)$ to $\int_0^t h[x(s)]\,ds$. Letting $\gamma \uparrow \infty$, we get from (5)

$$G(a,b) - \frac{G(a,\xi)G(\xi,b)}{G(\xi,\xi)} \tag{6}$$

which corresponds to

$$\int_0^\infty e^{-\alpha t} E_a \left[\exp\left[-\int_0^t h[x(s)]\,ds\right], t < m_\xi, x(t) \in db\right] dt \qquad a, b < \xi.$$

where $m_a(\eta) := \inf(s : \eta(s) = a)$ denotes the hitting time.

2.2 Random mappings and local time

As usual, a random mapping on the $\{1, \ldots, n\}$ is defined to be an element of the set F_n of all mappings $\varphi : \{1, \ldots, n\} \to \{1, \ldots, n\}$ equipped with the uniform distribution. It can be represented by its functional graph G_φ, i.e. the graph consisting of the nodes $1, 2, \ldots, n$ and of the edges $(i, \varphi(i))$, $i = 1, \ldots, n$. It is easy to see that each component of such a graph consists of exactly one cycle of length ≥ 1 each point of which is the root of a labeled tree (a so called Cayley tree). Thus for each point $x \in G_\varphi$ there exists a unique path connecting x with the next cyclic point. The length of this path is called the distance of x to the cycle. The set of all points at a fixed distance r from the cycle is often called the r-th stratum of φ.

Let $L_n(t)$ denote the number of nodes in the t-th stratum of a random mapping $\varphi \in F_n$. For noninteger t define

$$L_n(t) = (\lfloor t \rfloor + 1 - t)L_n(\lfloor t \rfloor) + (t - \lfloor t \rfloor)L_n(\lfloor t \rfloor + 1), \quad t \geq 0.$$

There is a lot of literature on random mappings, and interested readers should consult e.g. [16]. In the sequel we will need the following result from [7]:

Theorem 2.1 *The following limit theorem holds in $C[0,\infty)$:*

$$\frac{1}{\sqrt{n}} L_n(t\sqrt{n}) \xrightarrow{w} \frac{1}{2}\tau^+\left(\frac{t}{2}\right)$$

in $C[0,\infty)$, as $n \to \infty$.

By means of this theorem we will compute the multi-dimensional local time denstities: Let $a_n = n^{n-1}$ be the number of Cayley trees consisting of n nodes. Furthermore denote by $a^{(r_1 \cdots r_d)}_{k_1 \cdots k_d n}$ the number of all random mappings in F_n which have k_i nodes in stratum r_i, where $r_1 < \ldots < r_d$. Then by setting

$$\begin{aligned} b_0(z,u) &= u \\ b_{i+1}(z,u) &= z e^{b_i(z,u)} \end{aligned} \tag{7}$$

and

$$a(z) = \sum_{n \geq 0} a_n \frac{z^n}{n!}$$

and using standard combinatorial techniques (readers not familiar with these techniques may consult e.g. [10] and [8]) we can write down the generating function of these numbers in the form

$$\begin{aligned} g(z,u_1,\ldots,u_d) &= \sum_{k_1,\ldots,k_d,n\geq 0} a^{(r_1\cdots r_d)}_{k_1\cdots k_d n} u_1^{k_1}\cdots u_d^{k_d} z^n \\ &= \frac{1}{1-b_{r_1}(z,u_1 b_{h_{12}}(z,u_2 b_{h_{23}}(\cdots u_{d-1} b_{h_{d-1,d}}(z,u_d a(z))\cdots))} \end{aligned}$$

where $h_{ij} = r_j - r_i$.

The multi-dimensional density can be determined by evaluating the proper coefficient of $g(z,u_1,\ldots,u_d)$: If we set $r_i = \lfloor \rho_i\sqrt{n}\rfloor$ and $k_i = \lfloor y_i\sqrt{n}\rfloor$ and denote by $f_{\rho_1\cdots\rho_d}(y_1,\ldots,y_d)$ the joint density of
$\tau^+(\rho_1),\ldots,\tau^+(\rho_d)$, then by Theorem 2.1 we have

$$f_{\rho_1/2,\ldots,\rho_d/2}(2y_1,\ldots,2y_d) = 2^{-d}\lim_{n\to\infty}\frac{n^{d/2}n!}{n^n}[z^n u_1^{k_1}\cdots u_d^{k_d}]g(z,u_1,\ldots,u_d) \tag{8}$$

where the symbol $[z^n]f(z)$ denotes the coefficient of z^n in the power series expansion of $f(z)$.

3 Preliminary formulas

In this section, we define some auxiliary functions built on Kac's formula. Let (ρ_i) be a strictly monotonically increasing sequence of non-negative real numbers. Set

$$\varphi_d(\alpha) := \int_0^\infty e^{-\alpha t} E\left[e^{-\beta_1\sqrt{t}\tau^+(\rho_1/\sqrt{t})-\cdots-\beta_d\sqrt{t}\tau^+(\rho_d/\sqrt{t})}\right]\frac{dt}{\sqrt{2\pi t}}$$

and

$$\bar\varphi_d(\alpha) := \int_0^\infty [e^{-\alpha t}-1] E\left[e^{-\beta_1\sqrt{t}\tau^+(\rho_1/\sqrt{t})-\cdots-\beta_{d-1}\sqrt{t}\tau^+(\rho_{d-1}/\sqrt{t})}[t<m_{\rho_d}]\right]\frac{dt}{\sqrt{2\pi t}}.$$

where $m_.$ is related to the Brownian bridge of duration t.

Lemma 1 *We have*

$$\varphi_d(\alpha) = G_d^{**}(a,b)|_{a=b=0} \tag{9}$$

and

$$\bar\varphi_d(\alpha) = \bar G_d^{**}(a,b)|_{a=b=0} \tag{10}$$

where

$$\begin{aligned} G_d^{**}(a,b) &= G_d^*(a,b) - \beta_d\frac{G_d^*(a,-\rho_d)G_d^*(-\rho_d,b)}{1+\beta_d G_d^*(-\rho_d,-\rho_d)} \\ \bar G_d^{**}(a,b) &= \bar G_d^*(a,b) - \frac{\bar G_d^*(a,-\rho_d)\bar G_d^*(-\rho_d,b)}{\bar G_d^*(-\rho_d,-\rho_d)} \end{aligned} \tag{11}$$

with

$$G_d^*(a,b) = \begin{cases} G(a,b) - \beta_1\dfrac{G(a,\rho_1)G(\rho_1,b)}{1+\beta_1 G(\rho_1,\rho_1)} & \text{for } d=1 \\ G_{d-1}^{**}(a,b) - \beta_d\dfrac{G_{d-1}^{**}(a,\rho_d)G_{d-1}^{**}(\rho_d,b)}{1+\beta_d G_{d-1}^{**}(\rho_d,\rho_d)} & \text{for } d>1 \end{cases} \tag{12}$$

and

$$\bar{G}_d^*(a,b) = \begin{cases} G(a,b) - \dfrac{G(a,\rho_1)G(\rho_1,b)}{G(\rho_1,\rho_1)} & \text{for } d=1 \\ G_{d-1}^{**}(a,b) - \dfrac{G_{d-1}^{**}(a,\rho_d)G_{d-1}^{**}(\rho_d,b)}{G_{d-1}^{**}(\rho_d,\rho_d)} & \text{for } d>1 \end{cases} \tag{13}$$

and

$$G(a,b) = \frac{\exp\left(-\sqrt{2\alpha}|a-b|\right)}{\sqrt{2\alpha}}.$$

Proof: By (2) we have

$$G(a,b)db = \int_0^\infty e^{-\alpha t} E_a[x(t)\in db]dt.$$

Inserting $h\equiv 0$ and $f(y) = I_{db}(y)$ into Kac's formula and adding $\gamma t^+(t,\rho)$ as described before formula (6) yields

$$\begin{aligned} G_1(a,b)db &= \lim_{\gamma\uparrow\infty}\int_0^\infty e^{-\gamma t}E_a[\exp(-\gamma t^+(t,\rho)), x(t)\in db]dt \\ &= \int_0^\infty e^{-\alpha t}E_a[x(t)\in db, t<m_\rho]dt. \end{aligned}$$

Adding a local time $\beta t^+(t,\rho)$ to the exponent gives a modification of the Green function according to (5) and thus we have

$$G_2(a,b) = \int_0^\infty e^{-\alpha t}E_a[e^{-\beta t^+(t,\rho)}x(t)\in db]dt$$

and continuing in this way (considering $\rho_d, -\rho_d$) yields the recursion (12) and (11). For obtaining (13) we have to take into account that the d-th step we restrict to $[t<m_{\rho_d}]$ and $[t<m_{-\rho_d}]$ (in the way we got (5) and (6), i.e. by adding $\gamma t^+(t,\rho_d)$ and $\gamma t^+(t,-\rho_d)$ to the exponent and letting $\gamma\uparrow\infty$ instead of adding the local time $\beta_1 t^+(t,\rho_1)$.

□

Examples
Dimension 0
$\bar{\varphi}_1(\alpha) = \frac{Sh(\sqrt{2\alpha}x_1)}{\sqrt{2\alpha}Ch(\sqrt{2\alpha}x_1)}$: it is correct and corresponds to the Kolmogorov-Smirnov formula.
Dimension 1

$$\begin{aligned} \varphi_1(\alpha) - \bar{\varphi}_1(\alpha) &= \frac{2e^{2\sqrt{2\alpha}x_1}}{(1+e^{2\sqrt{2\alpha}x_1})(\sqrt{2\alpha}e^{2\sqrt{2\alpha}x_1} + \beta_1[1+e^{2\sqrt{2\alpha}x_1}])} \\ &= \frac{1}{Ch(\sqrt{2\alpha}x_1)[\sqrt{2\alpha}e^{\sqrt{2\alpha}x_1} + 2\beta_1 Ch(\sqrt{2\alpha}x_1)]} \end{aligned}$$

Inverting, this leads to (1)
Dimension 2

$$\varphi_2(\alpha) - \bar{\varphi}_2(\alpha) = \frac{\alpha}{2f_3^2[\beta_2 + \frac{\sqrt{\alpha}E^{2,1}}{\sqrt{2}Sh_{2,1}} + \frac{f_5}{f_3}]}$$

with

$$f_5 \quad := \quad -\alpha\frac{Ch_1}{Sh_{2,1}}$$

$$
\begin{aligned}
f_3 &= \sqrt{2}Ch_1 Sh_{2,1}[\beta_1 + \frac{\sqrt{\alpha}Ch_2}{\sqrt{2}Ch_1 Sh_{2,1}}] \\
Ch_1 &:= \cosh(\sqrt{2\alpha}\rho_1) \\
Sh_{2,1} &:= \sinh(\sqrt{2\alpha}(\rho_2 - \rho_1)) \\
E^{2,1} &:= e^{\sqrt{2\alpha}(\rho_2-\rho_1)}
\end{aligned}
$$

Inverting leads to

$$
\frac{\alpha}{4Ch_1^2 Sh_{2,1}^2} e^{\frac{-y_1\sqrt{\alpha}Ch_2}{\sqrt{2}Ch_1 Sh_{2,1}} - y_2 \frac{\sqrt{\alpha}E^{2,1}}{\sqrt{2}Sh_{2,1}}} \sqrt{\frac{y_1}{a_1}} I_1(2\sqrt{a_1 y_1})
$$

$$
a_1 = \frac{\alpha y_2}{2Sh_{2,1}^2}
$$

In order to get our density representations we have to invert the formulas. This is done by the following

Lemma 2 *Set $\mathcal{B}_d(t) := e^{-\beta_1\sqrt{t}\tau^+(\rho_1/\sqrt{t})\cdots-\beta_d\sqrt{t}\tau^+(\rho_d/\sqrt{t})}$. We obtain the following inversion formulas:*

$$
E[\mathcal{B}_d(1), 1 > m_{\rho_d}] = \frac{1}{\sqrt{2\pi i}} \int_S e^u[\Psi_d(u) - \bar{\Psi}_d(u)]\, du
$$

Proof: See [12, Section 4].

□

4 Multi-dimensional densities

We offer two different proofs of our results: the first one is based on some properties of G_d^{**}. The other one is based on Cauchy's formula applied to (8) and singularity analysis.

Lemma 3 *We have*

A. $\varphi_d(\alpha) - \bar{\varphi}_d(\alpha) = \Theta(d)$ *with*

$$
\Theta(d) := \frac{\alpha^{d-1}}{2[F_1(d)]^2[\beta_d + C_1(d) + C_2(d)D_2(d)/F_1(d)]}
$$

where

$$
\begin{aligned}
C_1(d) &= \sqrt{\frac{\alpha}{2}} E^{d,d-1}/Sh_{d,d-1} \\
C_2(d) &= -\frac{\alpha}{2Sh_{d,d-1}^2} \\
C_3(d) &= \sqrt{\frac{\alpha}{2}} \frac{Sh_{d,d-2}}{Sh_{d,d-1}Sh_{d-1,d-2}} \\
F_1(d) &= \beta_{d-1}D_2(d) + D_1(d) \\
D_2(d) &= (\beta_{d-2}D_2(d-1) + D_1(d-1))\sqrt{2}Sh_{d,d-1} \\
&= \sqrt{2}Sh_{d,d-1}F_1(d-1) \\
D_1(d) &= C_3(d)D_2(d) + C_2(d-1)D_2(d-1)\sqrt{2}Sh_{d,d-1} \\
&= \sqrt{2}Sh_{d,d-1}C_3(d)F_1(d-1) + 2Sh_{d,d-1}Sh_{d-1,d-2}C_2(d-1)F_1(d-2)
\end{aligned}
$$

The coefficient of $\beta_1 \cdots \beta_{d-2}$ in $D_2(d)$ equals $2^{(d-1)/2}Ch_1 \prod_{l=2}^d Sh_{l,l-1}$ or, equivalently, the initial values of $F_1(d)$ are given by

$$
F_1(1) = Ch_1 \qquad F_1(2) = \sqrt{2}Ch_1 Sh_{2,1}[\beta + \frac{\sqrt{2}Ch_2}{\sqrt{2}Ch_1 Sh_{2,1}}]
$$

where $Sh_{i,j} = \sinh(\sqrt{2\alpha}(\rho_i - \rho_j))$, $Ch_{i,j} = \cosh(\sqrt{2\alpha}(\rho_i - \rho_j))$, and $E^{i,j} = e^{\sqrt{2\alpha}(\rho_i-\rho_j)}$.

B.
$$\bar{\varphi}_d(\alpha) - \bar{\varphi}_{d-1}(\alpha) = \Theta(d-1) - \frac{\alpha^{d-1}}{2[F_1(d)]^2[C_1(d) + C_2(d)D_2(d)/F_1(d)]}$$

Once we have proved Lemma 3, it is now routine to derive the following Theorem: Part A is computed as in [12] in Part B, we use the transforms

$$\mathcal{L}_\alpha \left[\frac{1}{a}(1 - e^{ax})\right] = \frac{1}{\alpha(\alpha + a)}$$

Theorem 4.1 .

A. *The d-dimensional density is given by*

$$\begin{aligned} & f_{\rho_1 \cdots \rho_d}(y_1, y_2, \ldots, y_d; M > \rho_d) = \\ & \frac{1}{i\sqrt{2\pi}} \int_S \frac{e^\alpha \alpha^{d-1}}{2^d Ch_1^2 \prod_{l=2}^d Sh_{l,l-1}^2} \cdot \\ & \cdot \exp\left[-y_d C_1(d) - \sum_{l=2}^{d-1} y_l C_3(l+1) - y_1 \frac{\sqrt{\alpha} Ch_2}{\sqrt{2} Ch_1 Sh_{2,1}}\right] \cdot \\ & \cdot \prod_{l=1}^{d-1} \left[\sqrt{\frac{y_l}{a_l y_{l+1}}} I_1[2\sqrt{a_l y_l y_{l+1}}]\right] d\alpha \end{aligned} \tag{14}$$

with $M = \sup_{u \in [0,1]}[Y(u)|L = 1]$ *and* $a_l := -\frac{\alpha}{2Sh_{l+1,l}^2}$.

B. *The constraint densities* $f_{\rho_1 \cdots \rho_d}(y_1, \ldots, y_k; \rho_{k+1} > M > \rho_k)$ *are given by (14) where we replace* d *by* k *and* $-y_k C_1(k)$ *by* $-y_k C_3(k+1)$.

4.1 The proofs

4.1.1 Using some properties of G_d^{**}

Proof: *Proof of Lemma 3, Part A.* Actually, we will use the same notation as in the proof of [12, Lemma 3] and use auxiliary functions $D_3(d)$, $D_4(d)$, for which we will prove the following relations:

$$\begin{aligned} D_2(d) &= \beta_{d-2} D_4(d) + D_3(d) \\ \frac{D_1(d)}{D_2(d)} &= C_3(d) + C_4(d) D_4(d)/D_2(d) \\ C_4(d) &= C_2(d-1) \\ \frac{D_3(d)}{D_4(d)} &= \frac{D_1(d-1)}{D_2(d-1)} \\ D_4(d) &= D_2(d-1)\sqrt{2} Sh_{d,d-1} \end{aligned}$$

The coefficient of $\beta_1 \cdots \beta_{d-3}$ in $D_4(d) = 2^{(d-1)/2} Ch_1 \prod_{l=2}^d Sh_{l,l-1}$.

By (12) and (13) we have

$$\begin{aligned} G_d^{**}(a,b) - \bar{G}_d^{**}(a,b) &= \frac{2G_{d-1}^{**}(a,\rho_d) G_{d-1}^{**}(\rho_d, b)}{[1 + \beta_d G_{d-1}^{**}(\rho_d,\rho_d)] \hat{G}_{d-1}(\rho_d,\rho_d)} \\ &= \frac{\hat{G}_{d-1}(a,\rho_d)\hat{G}_{d-1}(\rho_d,b)}{2[\hat{G}_{d-1}(\rho_d,\rho_d)]^2[\beta_d + \frac{1}{\hat{G}_{d-1}(\rho_d,\rho_d)}]} \end{aligned} \tag{15}$$

where $\hat{G}_d(a,b) = G_d^{**}(a,b) - G_d^{**}(a,-b)$.

We have $\hat{G}_d(a,b) = \hat{G}(b,a), \hat{G}_d(a,b) = \hat{G}_{d-1}(a,b) - \beta_d \frac{\hat{G}_{d-1}(a,\rho_d)\hat{G}_{d-1}(\rho_d,b)}{1+\beta_d \hat{G}_{d-1}(\rho_d,\rho_d)}$ and $\hat{G}$ satisfies the 4 relations in the appendix of [12]. From now on we use $\hat{G}$ instead of G^{**}, the proof follows closely Sec. 5.4.1 of [12],

Proof: *Proof of Lemma 3, Part B.*

We must now analyze $\bar{G}_d^{**}(a,b) - \bar{G}_{d-1}^{**}(a,b)$

$$= G_{d-1}^{**}(a,b) - \bar{G}_{d-1}^{**}(a,b) - \frac{\hat{G}_{d-1}(a,\rho_d)\hat{G}_{d-1}(b,\rho_d)}{2\hat{G}_{d-1}(\rho_d,\rho_d)}$$

This first part clearly leads to $\Theta(d-1)$.

The second part gives

$$-\frac{\alpha^{d-1}}{2F_1^2[C_1(d) + C_2(d)D_2(d)/F_1(d)]}$$

□

4.1.2 Using the random mapping approach

Now we will use the results of section 2.2 in order to deduce Theorem 4.1. The proof runs in the following way: First we apply [6, Lemma 2.1] in order to get an asymptotic expansion of $g(z, u_1, \ldots, u_d)$. Then we will apply Cauchy's formula and singularity analysis in the sense of Flajolet and Odlyzko [9]. We will omit details like error estimates, since this works in a very similar way as in [12]. There is also another way to get a more rigorous proof via the random mapping approach: When we consider random mapping built of planted plane trees instead of Cayley trees. Since this can be viewed as a special case of constrained random mappings (see [3, 4, 11]), it is easy to see that Theorem 2.1 still holds (with a different scaling parameter of course: $\sqrt{2}$ instead of 2). Thus the explicit formulas [12, eq. 31 and eq. 32] can be used instead of the asymptotic ones below and the error estimates are much easier. But on the other hand, dealing with those explicit expressions is much more involved and does not provide any deeper insight.

We have

Lemma 4 *Assume that $|z - \frac{1}{e}| \to 0$ in such a way that $|a(z)| \le 1 + \mathcal{O}\left(\frac{1}{\sqrt{n}}\right)$, as $n \to \infty$, and $|u - a(z)| = \mathcal{O}\left(\frac{1}{\sqrt{n}}\right)$. Then we have the following asymptotic relation uniformly for $r = \mathcal{O}(\sqrt{n})$:*

$$b_r(z,u) = \frac{A_r - uB_r}{C_r - uD_r}\left(1 + \mathcal{O}\left((1-a(z))\right)\right)$$

where

$$\begin{aligned} A_r &= \left(1 + \frac{a(z)}{2}\frac{1-a(z)^r}{1-a(z)} - a(z)^r\right)a(z) \\ B_r &= \frac{a(z)}{2}\frac{1-a(z)^r}{1-a(z)} - a(z)^r \\ C_r &= 1 + \frac{a(z)}{2}\frac{1-a(z)^r}{1-a(z)} \\ D_r &= \frac{1}{2}\frac{1-a(z)^r}{1-a(z)} \end{aligned}$$

Proof: Set $\varphi(t) = e^t$, $\sigma = 1$, and $\tau = 1$ in [6, Lemma 2.1]. Then the above form can be obtained by elementary algebra.

□

Using this expansion we obtain after a straight forward calculation the following formula: $[u_1^{k_1}\cdots u_d^{k_d}]g(z,u_1,\ldots,u_d) = [z^n]F(z)$ with

$$\begin{aligned} F(z) &= \left(\frac{C_{r_1}}{C_{r_1}-A_{r_1}} - \frac{D_{r_1}}{D_{r_1}-B_{r_1}}\right)\left(\frac{D_{r_1}-B_{r_1}}{C_{r_1}-A_{r_1}}\right)^{k_1}\left(\frac{A_{h_{12}}}{C_{h_{12}}}\right)^{k_1}\left(\frac{D_{h_{d-1,d}}}{C_{h_{d-1,d}}}\right)^{k_d} S_d \\ &\quad \times \prod_{l=2}^{d-1}\left[\left(\frac{D_{h_{l-1,l}}}{C_{h_{l-1,l}}}\right)^{k_l}\left(\frac{A_{h_{l,l+1}}}{C_{h_{l,l+1}}}\right)^{k_l} S_l\right], \end{aligned} \tag{16}$$

where

$$S_l = \sum_{i=0}^{\min(k_{l-1},k_l)-1}\binom{k_{l-1}}{i+1}\binom{k_l-1}{i}\left(1-\frac{B_{h_{l-1,l}}C_{h_{l-1,l}}}{A_{h_{l-1,l}}D_{h_{l-1,l}}}\right)^{i+1}.$$

Now, in order to calculate this coefficient we use Cauchy's integral formula choosing a truncated line normal to the real axis and complemented by a circular arc as integration path. To be precise, we integrate along $\Gamma = \gamma' \cup \Gamma'$ given by

$$\begin{aligned} \gamma' &= \left\{z : z = \frac{1}{e}\left(1 - \frac{1+it}{n}\right) \text{ and } |t| \le \sqrt{2n+1}\right\} \\ \Gamma' &= \left\{z : |z| = \frac{1}{e} \text{ and } \arctan\frac{\sqrt{2n+1}}{n-1} \le |\arg z| \le \pi\right\} \end{aligned}$$

On γ' we substitute $z = \frac{1}{e}\left(1-\frac{\alpha}{n}\right)$. Now using the well known expansion for the tree function (see e.g. [19]) on γ'

$$a(z) \sim 1 - \sqrt{\frac{2\alpha}{n}}$$

we obtain the asymptotic relations

$$\frac{C_{r_1}}{C_{r_1}-A_{r_1}} - \frac{D_{r_1}}{D_{r_1}-B_{r_1}} \sim \frac{1}{\cosh^2\left(\rho_1\sqrt{\alpha/2}\right)} \tag{17}$$

$$\left(\frac{D_{r_1}-B_{r_1}}{C_{r_1}-A_{r_1}}\right)^{k_1} \sim \exp\left(-y_1\sqrt{2\alpha}\frac{\sinh\left(\rho_1\sqrt{\alpha/2}\right)}{\cosh\left(\rho_1\sqrt{\alpha/2}\right)}\right) \tag{18}$$

$$\left(\frac{A_{h_{l,l+1}}}{C_{h_{l,l+1}}}\right)^{k}_{l} \sim \exp\left(-\frac{y_l\sqrt{2\alpha}\cosh\left((\rho_{l+1}-\rho_l)\sqrt{\alpha/2}\right)}{\sinh\left((\rho_{l+1}-\rho_l)\sqrt{\alpha/2}\right)}\right) \tag{19}$$

$$\left(\frac{D_{h_{l,l+1}}}{C_{h_{l,l+1}}}\right)^{k}_{l+1} \sim \exp\left(-\frac{y_l\sqrt{2\alpha}\cosh\left((\rho_{l+1}-\rho_l)\sqrt{\alpha/2}\right)}{\sinh\left((\rho_{l+1}-\rho_l)\sqrt{\alpha/2}\right)}\right) \tag{20}$$

The sum in S_l can be approximated with the help of Stirling's formula and by extending the range of summation to infinity (for details see [12]). In this way we get

$$S_l \sim \frac{2\alpha}{\sinh^2\left((\rho_{l+1}-\rho_l)\sqrt{\alpha/2}\right)}\sqrt{\frac{y_l\sinh^2\left((\rho_{l+1}-\rho_l)\sqrt{\alpha/2}\right)}{2y_{l+1}\alpha}}\, I_1\left(2\sqrt{\frac{2\alpha y_l y_{l+1}}{\sinh^2\left((\rho_{l+1}-\rho_l)\sqrt{\alpha/2}\right)}}\right) \tag{21}$$

and inserting (17)–(21) into (16) and applying (8) yields (4.1) as desired.

Proof of Part B: This part is immediate: One has just to compute the coefficients

$$[z^n u_1^0 \cdots u_d^0]g(z,u_1,\ldots,u_d)$$

$$[z^n u_1^{k_1} u_2^0 \cdots u_d^0] g(z, u_1, \ldots, u_d)$$
$$\vdots$$
$$[z^n u_1^{k_1} \cdots u_{d-1}^{k_{d-1}} u_d^0] g(z, u_1, \ldots, u_d)$$

which is an easy exercise.

□

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