Reflected Brownian Bridge area conditioned on its local time at the origin

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Abstract

Using some properties of the Airy functions, we analyze the reflected Brownian Bridge area W_b conditioned on its local time b at the origin. We give a closed form expression of the Laplace transform of W_b, a recurrence equation for the moments, leading to an efficient computation algorithm and an asymptotic form for the density $f(x, b)$ of W_b for $x \downarrow 0$.

1 Introduction

Throughout this paper, the standard Brownian motion (BM) will be denoted by $x(t)$. Other classical BM are the reflected BM: $x^+(t) := |x(t)|$, the Brownian Bridge (BB) on $[0, 1]$: $B(t)$, the reflected BB on $[0, 1]$: $B^+(t)$, the Brownian Excursion: $e(t)$. The local time of $x(t)$ at a, will be denoted by $t^+(t, a)$. Following Janson [9, remark 2.3], we define $W_b := \int_0^1 B^+(t)dt$ (area of the reflected BB), conditioned on having a local time at the origin equal to b.

In [9], W_b appears as the limit law for $m^{-3/2}D_{m,m-b\sqrt{m}}$, where $D_{m,m-b\sqrt{m}}$ denotes the total displacement for a hashing table with m places and $b\sqrt{m}$ empty places, using linear probing. Previously, the case $b \downarrow 0$ was investigated by Flajolet, Viola and Poblete [5]. They obtained the Airy distribution (the distribution of the Brownian excursion area $\int_0^1 e(t)\ dt$) as the limit law of $m^{-3/2}D_{m,m-1}$, so we propose to call the distribution of W_b the "Generalized Airy Distribution". As a consequence of [2], we also have

$$\begin{aligned} W_b &\overset{law}{=} \int_0^1 e(t) - bt - \min_{0\le s\le t} (e(s) - bs)\ dt \\ &\overset{law}{=} \int_0^1 B(t) - bt - \min_{-\infty\le s\le t} (B(s) - bs)\ dt \end{aligned}$$

where the Brownian bridge B is periodically extended to the whole line: the case $b = 0$ leads again to $\int_0^1 e(t)\ dt$, using a Theorem of Vervaat for the second description. The random variable W_b can also be seen as the limit law for the path length of a random forest with $b\sqrt{m}$ trees and m nodes or leaves, with a similar scaling.

A simple description of the law of W_b still lacks. A first attempt was given by Janson in a talk presented at AOFA'2000 and described in [9], where the moments of W_b, i.e.

$$\psi_k(b) := \mathbb{E}[W_b^k],$$

are given by the following Theorem[9, Th. 3.3]:

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Theorem 1.1

$$\psi_k(b) \quad = \quad k! \sum_{j=1}^{k} \left(\sum_{k_1,\ldots,k_j \geq 1,\ \Sigma k_i = k} \prod_{i=1}^{j} \omega_{k_j} \right) \frac{b^{j-1}}{j!} q_{3k-j-2}(b),$$

in which:

$$q_r(b) \quad := \quad \int_0^{+\infty} \frac{x^r}{r!} e^{-bx-x^2/2}\, dx,$$

$$\frac{Ai'(z)}{Ai(z)} \quad \underset{z\to+\infty}{\sim} \quad \sum_{k=0}^{+\infty} \omega_k \frac{(-1)^k z^{-(3k-1)/2}}{2^k}.$$

Here Ai denotes the classical Airy function

$$\mathrm{Ai}(z) \quad := \quad \frac{1}{\pi} \int_0^{+\infty} \cos\left(\frac{1}{3}t^3 + zt\right)\, dt,$$

$$:= \quad \frac{1}{\pi}\left(\frac{z}{3}\right)^{1/2} K_{1/3}\left(\frac{2z^{3/2}}{3}\right).$$

and $K_{1/3}$ denotes the Bessel function of order 1/3.

Another expression is given by Knight [10]. He obtains

Theorem 1.2

$$\psi_k(b) = k! G_k(b),$$

with

$$G_n(s) = \exp(s^2/16) \sum_{j=n}^{3n-2} d_{j,n} D_{-j}(s/2),$$

where $D_j(x)$ is the parabolic cylinder function and $d_{j,n}$ is given by a recurrence equation.

In Section 3, we provide a more efficient recurrence formula for the computation of the moments. Our other contributions are a closed form expression of the Laplace transform $\Theta(z,b)$ of W_b,

Theorem 1.3

$$\Theta(z,b) = \mathbb{E}[e^{-zW_b}] = \frac{-z^{1/3}e^{b^2/2}}{i\,2^{1/6}\sqrt{\pi}} \int_{-i\infty}^{i\infty} e^{bz^{1/3}2^{\cdot} Ai'(u)/Ai(u)} [Ai'(u)/Ai(u)]' e^{uz^{2/3}/2^{\cdot}}\, du.$$

and two asymptotic forms for the density $f(x,b)$ of W_b for $x \downarrow 0$:

Theorem 1.4

$$f(x,b) \underset{x\downarrow 0}{\sim} e^{\mu_2/x^2} \frac{\sqrt{2}}{\sqrt{\pi}} \left[\frac{3^{1/4}\alpha_1^{9/4}}{9x^{11/4}b^{3/4}} - \frac{3^{3/4}\alpha_1^{3/4}}{3x^{9/4}b^{1/4}} + \frac{b^{1/4}3^{1/4}(27+16\alpha_1^3)}{x^{7/4}\alpha_1^{3/4}} + \mathcal{O}\left(\frac{1}{x^{5/4}}\right)\right].$$

where μ_2 is given in (31) and α_1 is the absolute value of the first zero of $Ai(u)$.

Theorem 1.5 *For small b,*

$$f(x,b) \underset{x\downarrow 0}{\sim} f_0(x) + bf_1(x) + \mathcal{O}(b^2),$$

where f_0 and f_1 are given in (34) and (35).

As in [12], [11] and [4], where the law of $X := \int_0^1 e(t)dt$ is investigated, the Airy function properties play an important rôle. The paper is organized as follows: in Section 2 we give the basic results needed in the sequel. We also relate the moments of W_b to an asymptotic form of the Airy function. In Section 3, we give a recurrence formula for the moments. Section 4 is devoted to the proofs of Theorems 1.2 and 1.3. Section 5 concludes the paper. MAPLE was again of great help in some complicated computations.

2 Basic results

In this section, we show how to relate the moments expressions to asympotic forms of Ai and $Ai\prime$. This leads to easily computing the moments in terms of Gaussian distribution function iterates. Let us first prove the following Lemma.

Lemma 2.1

$$\int_0^\infty e^{-\alpha t}\,\mathbb{E}_0\left[exp\left[-\int_0^t x^+(u)du-\delta t^+(t,0)\right]\Big|\, x(t)=0\right]\frac{dt}{\sqrt{2\pi t}}=\left[\delta-\frac{2^\cdot Ai'(2^\cdot\alpha)}{Ai(2^\cdot\alpha)}\right]^{-1}. \tag{1}$$

where $2^\cdot := 2^{1/3}$, Ai is the classical Airy function.
Set $G(\eta) := 2^\cdot\eta^{1/3}\frac{Ai'(2^\cdot\eta^{-2/3})}{Ai(2^\cdot\eta^{-2/3})}, G_0 := G(0)$. Then

$$\int_0^\infty e^{-1/(2s)}e^{-ws}\mathbb{E}\left[e^{-\eta w^{3/2}s^{3/2}W_b}-1\right]\frac{ds}{s^{3/2}\sqrt{2\pi}}=e^{\sqrt{w}G(\eta)}-e^{\sqrt{w}G_0}=e^{\sqrt{w}G_0}\sum_1^\infty\frac{[\sqrt{w}[G(\eta)-G_0]]^i}{i!}. \tag{2}$$

Proof We know that, if we define

$$u(a):=\mathbb{E}_a\int_0^\infty e^{-\alpha t}exp\left[-\int_0^t h[x(s)]ds\right]f(x(t))dt$$

where $\alpha>0$, h is a piecewise continuous non-negative function, f is continuous, then $u(a)=\int G(a,b)f(b)db$ where the Green function G can be computed with the Kac formula. Moreover(see Ito and McKean [8] and Louchard [12]), if we add $\gamma t^+(t,\xi)$ to $\int_0^t h[x(s)]ds$, the modified Green function is given by

$$G^*(a,b)=G(a,b)-\gamma\frac{G(a,\xi)G(\xi,b)}{1+\gamma G(\xi,\xi)}. \tag{3}$$

Using (3), we can prove (1) (see [12], equ(30)). Now we need to find the distribution of the BB local time at the origin. We know that $G(0,0)=\mathcal{L}(p(t,0,0))=1/\sqrt{\cdot}$ where $\mathcal{L}$ is the Laplace transform, $p(t,x,y)$ is the classical BM density and $\sqrt{\cdot}{:=}\sqrt{2\alpha}$ (to simplify the notation). By (3),

$$\mathbb{E}_0\int_0^\infty e^{-\alpha t}\left[e^{-\beta t^+(t,0)},x(t)\in d0\right]dt=\left[\frac{1}{\sqrt{\cdot}}-\frac{\beta(\frac{1}{\sqrt{\cdot}})^2}{1+\beta/\sqrt{\cdot}}\right]d0=\frac{1}{\beta+\sqrt{\cdot}}d0.$$

This is also equivalent to

$$\mathbb{E}_0\int_0^\infty e^{-\alpha t}\left[e^{-\beta t^+(t,0)}\Big|\, x(t)=0\right]\frac{1}{\sqrt{2\pi t}}dt=\frac{1}{\beta+\sqrt{\cdot}}.$$

Inverting on β leads to

$$\int_0^\infty e^{-\alpha t}Pr_0[[t^+(t,0)|\,x(t)=0]\in db]\frac{1}{\sqrt{2\pi t}}dt=e^{-\sqrt{\cdot}b}db,$$

or

$$Pr_0[[t^+(t,0)|\,x(t)=0]\in db]=be^{-b^2/(2t)}db/t,$$

which is nothing but the Rayleigh density. This can also be derived from Gittenberger and Louchard, [7].

Next we analyze the reflected BB area, conditioned on the local time at 0. By standard scaling, we know that $[\int_0^t x^+(u)du\Big|\,x(t)=0]\stackrel{\mathcal{D}}{\equiv}t^{3/2}W_b$ and $t^+(t,0)\stackrel{\mathcal{D}}{\equiv}\sqrt{t}t^+(1,0)$. Equ. (1) leads to

$$\mathbb{E}_0\int_0^\infty dt\,e^{-\alpha t}\int_0^\infty db\,e^{-t^{3/2}W_b}be^{-b^2/2}e^{-\delta\sqrt{t}b}\frac{1}{\sqrt{2\pi t}}=[\delta-2^\cdot\Lambda(\alpha)]^{-1},$$

where $\Lambda(\alpha) := \frac{Ai'(2^{\cdot}\alpha)}{Ai(2^{\cdot}\alpha)}$.

Set $v := \sqrt{t}b$. This leads to

$$\int_0^\infty db \int_0^\infty dv\, e^{-\alpha v^2/b^2}\mathbb{E}\left[e^{-v^3/b^3 W_b}\right] e^{-b^2/2} e^{-\delta v}\frac{2}{\sqrt{2\pi}} = [\delta - 2^{\cdot}\Lambda(\alpha)]^{-1},$$

and inverting on δ, we obtain

$$\int_0^\infty db\, e^{-b^2/2}e^{-\alpha v^2/b^2}\mathbb{E}\left[e^{-v^3/b^3 W_b}\right]\frac{2}{\sqrt{2\pi}} = e^{v2^{\cdot}\Lambda(\alpha)}. \tag{4}$$

Set $s := 1/b^2, w = \alpha v^2$ and $\eta = \alpha^{-3/2}$. Equ. (4) becomes

$$\int_0^\infty e^{-1/(2s)}e^{-ws}\mathbb{E}\left[e^{-\eta w^{3/2}s^{3/2}W_b}\right]\frac{ds}{s^{3/2}\sqrt{2\pi}} = e^{\sqrt{w}G(\eta)}, \tag{5}$$

where $G(\eta) := \frac{2^{\cdot}\Lambda(\alpha)}{\sqrt{\alpha}}$. Set $G_0 := G(0)$. Equ. (5) leads, with $b = s^{-1/2}$, to (2). ■

Expanding both sides of (2) w.r.t η , this gives

$$\int_0^\infty e^{-1/(2s)}e^{-ws}(-1)^k s^{3k/2}\psi_k(b)\frac{ds}{s^{3/2}\sqrt{2\pi}k!} = [\eta^k]\frac{e^{\sqrt{w}G_0}}{w^{3k/2}}\sum_1^\infty\frac{[\sqrt{w}[G(\eta)-G_0]]^i}{i!}, \quad s = 1/b^2. \tag{6}$$

The LHS of (6), $L_1(k)$ say, is a Laplace transform, the RHS, $R_1(k)$, say, can be computed as follows. Abramowitz and Stegun [1] give us((10.4.59) and (10.4.61)):

$$\begin{aligned} Ai(z) &\sim \frac{1}{2\sqrt{\pi}}z^{-1/4}e^{-\zeta}\sum_0^\infty(-1)^k c_k\zeta^{-k}, \quad |z|\text{large}, \quad |arg(z)| < \pi, \\ Ai'(z) &\sim -\frac{1}{2\sqrt{\pi}}z^{1/4}e^{-\zeta}\sum_0^\infty(-1)^k d_k\zeta^{-k}, \end{aligned}$$

where

$$\begin{aligned} c_0 = 1, c_k = \varphi_k/(54^k k!), &\quad \text{with} \quad \varphi_k := \Gamma(3k+1/2)/\Gamma(k+1/2), \\ d_0 = 1, d_k &= -\frac{6k+1}{6k-1}c_k, \\ \zeta &= \frac{2}{3}z^{3/2}. \end{aligned}$$

We obtain

$$G(\eta) \sim -\sqrt{2}\sum_0^\infty(-1)^j d_j\left[\frac{3\eta}{2^{3/2}}\right]^j \Big/ \sum_0^\infty(-1)^j c_j\left[\frac{3\eta}{2^{3/2}}\right]^j.$$

Hence $G_0 = -\sqrt{2}$ and

$$G(\eta) \sim -\sqrt{2} - \eta/4 + 5/64\sqrt{2}\eta^2 - 15/128\eta^3 + 1105/8192\sqrt{2}\eta^4 - 1695/4096\eta^5 + \ldots$$

This leads with (6) to

$$R_1(k) = [\eta^k]\frac{e^{-\sqrt{2w}}}{w^{3/2k}}R(\eta),$$

with

$$\begin{aligned} R(\eta) = &-1/4w^{1/2}\eta + (5/64w^{1/2}\sqrt{2} + 1/32w)\eta^2 + (-15/128w^{1/2} - 5/256w\sqrt{2} - 1/384w^{3/2})\eta^3 \\ &+ (1105/8192w^{1/2}\sqrt{2} + 145/4096w + 5/2048w^{3/2}\sqrt{2} + 1/6144w^2)\eta^4 + \ldots. \end{aligned}$$

For instance

$$
\begin{aligned}
R_1(1) &= -e^{-\sqrt{2w}}/(4w),\\
R_1(2) &= e^{-\sqrt{2w}}(5/64\sqrt{2}/w^{5/2}+1/(32w^2)),\\
R_1(3) &= e^{-\sqrt{2w}}(-15/(128w^4)-5/256\sqrt{2}/w^{7/2}-1/(384w^3)),\\
R_1(4) &= e^{-\sqrt{2w}}(1105/8192\sqrt{2}/w^{11/2}+145/(4096w^5)+5/2048\sqrt{2}/w^{9/2}+1/(6144w^4)).
\end{aligned}
$$

Clearly, we need to invert Laplace transforms of the form $e^{-\sqrt{2w}}/w^{(j+1)/2}$. But we now prove the following lemma:

Lemma 2.2 *Set $\Phi^{(1)}(x)=\Phi(x)$ (classical Gaussian distribution function) and $\Phi^{(j+1)}(x):=\int_{-\infty}^{x}\Phi^{(j)}(u)du$. Then*

$$\int_0^\infty \Phi^{(j)}(-b)e^{-ws}\frac{(2s)^{(j+1)/2}}{s}ds=\frac{e^{-\sqrt{2w}}}{w^{(j+1)/2}},\quad j\geq 1 \tag{7}$$

where $b=1/\sqrt{s}$.

Proof Let us first check (7) for $j=1$. It is well known that

$$\int_0^\infty e^{-ws}e^{-1/(2s)}\frac{ds}{\sqrt{2\pi}s^{3/2}}=e^{-\sqrt{2w}}.$$

Set $s=1/b^2$. This gives

$$2\int_0^\infty e^{-w/b^2}e^{-b^2/2}\frac{db}{\sqrt{2\pi}}=e^{-\sqrt{2w}},$$

and, by parts,

$$\int_0^\infty \Phi(-b)e^{-w/b^2}4\frac{db}{b^3}=e^{-\sqrt{2w}}/w,$$

or

$$\int_0^\infty \Phi(-b)e^{-ws}2ds=e^{-\sqrt{2w}}/w.$$

Actually, we could even start the induction with $j=0$, using the density ϕ for $\Phi^{(0)}(-b)$. Let us now proceed by induction. Assume that (7) is true for some j. This gives

$$\int_0^\infty \Phi^{(j)}(-b)e^{-w/b^2}2^{(j+1)/2}\frac{2db}{b^{j+2}}=e^{-\sqrt{2w}}/w^{(j+1)/2},$$

and, by parts,

$$\int_0^\infty \Phi^{(j+1)}(-b)e^{-w/b^2}2^{(j+1)/2}[2w/b^3-(j+2)/b]\frac{w^{j/2}}{\sqrt{2}}\frac{2db}{b^{j+2}}=e^{-\sqrt{2w}}/\sqrt{2w}. \tag{8}$$

But proving (7) for $j+1$ amounts to prove

$$\int_0^\infty \Phi^{(j+1)}(-b)e^{-w/b^2}2^{(j+2)/2}w^{(j+2)/2}\frac{2db}{b^{j+3}}=e^{-\sqrt{2w}},$$

or, differentiating w.r t. w

$$\int_0^\infty \Phi^{(j+1)}(-b)e^{-w/b^2}2^{(j+2)/2}w^{(j+2)/2}[1/b^2-(j+2)/(2w)]\frac{2db}{b^{j+3}}=e^{-\sqrt{2w}}/\sqrt{2w},$$

which is exactly (8). We just integrate (8) w.r.t w on $[w..\infty]$. ■

Let us remark that $\Phi^{(j)}(x)$ can be expressed in term of $\Phi(x)$ and $e^{-x^2/2}/\sqrt{2\pi}$, we obtain

$$\begin{aligned}
\Phi^{(2)}(x) &= x\Phi(x) + e^{-x^2/2}/\sqrt{2\pi}, \\
\Phi^{(3)}(x) &= (x^2/2+1/2)\Phi(x) + x/2e^{-x^2/2}/\sqrt{2\pi}, \\
\Phi^{(4)}(x) &= (x^3/6+x/2)\Phi(x) + (x^2/6+1/3)e^{-x^2/2}/\sqrt{2\pi}.
\end{aligned}$$

In the next section, we will provide an efficient procedure to compute $\Phi^{(j)}(x)$.

Inverting (6) is now trivial. This gives

$$\begin{aligned}
\psi_1(b) &= 1/2e^{b^2/2}\sqrt{2\pi}\Phi(-b), \\
\psi_2(b) &= -1/12b(b^2+6)e^{b^2/2}\sqrt{2\pi}\Phi(-b) + 1/12(b^2+5), \\
\psi_3(b) &= 1/192(b^6+21b^4+93b^2+45)e^{b^2/2}\sqrt{2\pi}\Phi(-b) - 1/192b(b^4+20b^2+75), \\
\psi_4(b) &= -1/60480b(11b^8+495b^6+6615b^4+28665b^2+28350)e^{b^2/2}\sqrt{2\pi}\Phi(-b) \\
&\quad +1/60480(11b^8+484b^6+6153b^4+23370b^2+13260).
\end{aligned}$$

The first 2 expressions are given in Janson [9], cor 3.4.
Letting $b \downarrow 0$, we obtain of course the well-known moments of the Brownian excursion area (see[11]).

3 Recurrence formulae

In this section, we provide several efficient algorithms to compute $\Phi^{(j)}(x)$ and $\psi_k(b)$. Our results can be summarized in the following proposition.

Proposition 3.1

$$\Phi^{(k)}(z) = p_1(k,z)\Phi(z) + p_2(k,z)e^{-z^2/2}/\sqrt{2\pi},$$

where $p_1(k,z)$ is polynomial of degree $k-1$, $p_2(k,z)$ is polynomial of degree $k-2$. $p_1(k,z)$ and $p_2(k,z)$ are given by a set of recurrence equations explicitly detailed in (9) ... (11) below.
Define Z_j by

$$(-1)^j\frac{Z_j}{j!} = [\eta^j]\frac{1}{w^{3j/2}}\sum_1^\infty \frac{[\sqrt{w}[G(\eta)-G_0]]^i}{i!}.$$

Then $\psi_j(b)$ is equivalent to $Z_j b^{3j}\sqrt{2\pi}e^{b^2/2}/b^3$, where we use the transfer rule:

$$\frac{1}{w^{(l+1)/2}} \rightarrow \frac{\Phi^{(l)}(-b)}{b^{l+1}}b^2 2^{(l+1)/2}.$$

A recurrence for Z_k in terms of $Z_1 \dots Z_{k-1}$ is given, with $z = 1/\sqrt{w}$, by $L_2(k) = R_2(k)$ where

$$\begin{aligned}
L_2(k) &= \sum_{l=1}^k C[k,k-l](-1)^l Z_l z^{-3l}/l!, \\
R_2(k) &= \sum_{i=1}^k \frac{(-\sqrt{2})^i}{i!}z^{-i}B[k,i],
\end{aligned}$$

and C, B are given by (13) ... (16) below.

Let us first compute $\int_{-\infty}^z x^j e^{-x^2/2}/\sqrt{2\pi}dx$. By parts, this gives

- j odd: $-e^{-z^2/2}/\sqrt{2\pi}\sum_{l=0}^{(j-1)/2} z^{j-1-2l}(j)_l$, where we use a generalized Pochammer symbol: $(j)_l := \prod_{u=0}^{l-1}(j-1-2u)$,

- j even: $-e^{-z^2/2}/\sqrt{2\pi}\sum_{l=0}^{j/2-1} z^{j-1-2l}(j)_l + (j)_{j/2}\Phi(z)$.

Also
$$\int_{-\infty}^{z} x^j\Phi(x)dx = \frac{z^{j+1}}{j+1}\Phi(z) - \int_{-\infty}^{z}\frac{x^{j+1}}{j+1}e^{-x^2/2}/\sqrt{2\pi}dx.$$

We can now compute the matrices $P_1[1..\infty, 0..\infty]$ and $P_2[1..\infty, 0..\infty]$ giving the coefficients of p_1 and p_2. Omitting the (tedious) details, this leads to the following algorithm. First we set P_1 and P_2 to 0. We set $P_1[1,0] := 1$. For $k = 1..\infty$, we compute

$$\begin{aligned} P_1[k+1,j] &:= P_1[k,j-1]/j, j = 1..k, \qquad (9)\\ P_2[k+1,j] &:= \sum_{l=0}^{\lfloor (k-1-j)/2\rfloor} P_1[k,j+2l]/(j+2l+1)(j+2l+1)_l \\ &- \sum_{l=0}^{\lfloor (k-3-j)/2\rfloor} P_2[k,j+2l+1](j+2l+1)_l, j = 0..k-1, \qquad (10)\\ P_1[k+1,0] &:= -\sum_{l=1,\,by\,2}^{k-1} P_1[k,l]/(l+1)(l+1)_{(l+1)/2} + \sum_{l=0,\,by\,2}^{k-2} P_2[k,l](l)_{l/2}, \end{aligned}$$

which leads to
$$P_1[k,0] = 0, \text{ for } k \text{ even and } P_1[2j+1,0] = \frac{1}{2^j j!}. \qquad (11)$$

This gives
$$P_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0\\ 1/2 & 0 & 1/2 & 0 & 0 & 0 & 0 & 0\\ 0 & 1/2 & 0 & 1/6 & 0 & 0 & 0 & 0\\ 1/8 & 0 & 1/4 & 0 & 1/24 & 0 & 0 & 0\\ 0 & 1/8 & 0 & 1/12 & 0 & 1/120 & 0 & 0\\ 1/48 & 0 & 1/16 & 0 & 1/48 & 0 & 1/720 & 0 \end{pmatrix}$$

$$P_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 1/2 & 0 & 0 & 0 & 0 & 0 & 0\\ 1/3 & 0 & 1/6 & 0 & 0 & 0 & 0 & 0\\ 0 & 5/24 & 0 & 1/24 & 0 & 0 & 0 & 0\\ 1/15 & 0 & 3/40 & 0 & 1/120 & 0 & 0 & 0\\ 0 & 11/240 & 0 & 7/360 & 0 & 1/720 & 0 & 0 \end{pmatrix}$$

Let us now turn to $\psi.(b)$. If we define Z_j by (see (6)),
$$(-1)^j\frac{Z_j}{j!} = [\eta^j]\frac{1}{w^{3j/2}}\sum_{1}^{\infty}\frac{[\sqrt{w}[G(\eta)-G_0]]^i}{i!},$$

we see that $\psi_j(b)$ is equivalent to $Z_j b^{3j}\sqrt{2\pi}e^{b^2/2}/b^3$, where we use, with (7), the mechanical transfer rule:
$$\frac{1}{w^{(l+1)/2}} \to \frac{\Phi^{(l)}(-b)}{b^{l+1}}b^2 2^{(l+1)/2}.$$

To construct an efficient recurrence formula for Z_j, let us consider
$$S_2 := \sum_{i=1}^{k}(-1)^i\frac{Z_i}{i!}w^{3i/2}\eta^i.$$

This is also given by

$$S_3 := \sum_{i=1}^{k} \eta^i [\eta^i] \sum_{j=1}^{k} \left[\sum_{l=1}^{k} (-1)^l (d_l - c_l) \left[\frac{3\eta}{2^{3/2}}\right]^l \Big/ \sum_{l=0}^{k} (-1)^l c_l \left[\frac{3\eta}{2^{3/2}}\right]^l \right]^j \frac{(-\sqrt{2w})^j}{j!} .$$

Now, to simplify the expressions, we set $z = 1/\sqrt{w}$. S_3 becomes

$$S_3 = \sum_{i=1}^{k} \eta^i [\eta^i] \sum_{j=1}^{k} \frac{1}{\left[\sum_{l=0}^{k} (-1)^l c_l \left[\frac{3\eta}{2^{3/2}}\right]^l\right]^k} \left(\frac{-\sqrt{2}}{z}\right)^j \frac{1}{j!} \left[\sum_{l=1}^{k} (-1)^l (d_l - c_l) \left[\frac{3\eta}{2^{3/2}}\right]^l \right]^j \left[\sum_{l=0}^{k} (-1)^l c_l \left[\frac{3\eta}{2^{3/2}}\right]^l \right]^{k-j} .$$

Now we use the equivalence

$$[\eta^k] S_2 \left[\sum_{l=0}^{k} (-1)^l c_l \left[\frac{3\eta}{2^{3/2}}\right]^l \right]^k = [\eta^k] \sum_{j=1}^{k} \left(\frac{-\sqrt{2}}{z}\right)^j \frac{1}{j!} \left[\sum_{l=1}^{k} (-1)^l (d_l - c_l) \left[\frac{3\eta}{2^{3/2}}\right]^l \right]^j \left[\sum_{l=0}^{k} (-1)^l c_l \left[\frac{3\eta}{2^{3/2}}\right]^l \right]^{k-j} . \tag{12}$$

The LHS of (12), $L_2(k)$, say is given by

$$L_2(k) = \sum_{l=1}^{k} C[k, k-l] (-1)^l Z_l z^{-3l} / l!,$$

where

$$C[k,j] := [\eta^j] \left[\sum_{l=0}^{\infty} (-1)^l c_l \left[\frac{3\eta}{2^{3/2}}\right]^l \right]^k .$$

$C[0..\infty, 0..\infty]$ can be constructed as follows:

$$C[0,0] = 1, C[0,j] = 0, j = 1..\infty, \tag{13}$$

$$C[1,j] = c_j \left[\frac{3}{2^{3/2}}\right]^j (-1)^j, j = 0..\infty, \tag{14}$$

$$C[i+1,j] = \sum_{0}^{j} C[i, j-l] C[1,l], i \geq 1, j = 0..\infty. \tag{15}$$

This gives

$$C = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & -5/(24\, 2^{3/2}) & 385/(1152\, 2^3) & -85085/(82944\, 2^{9/2}) \\ 1 & -5/(12\, 2^{3/2}) & 205/(288\, 2^3) & -22715/(10368\, 2^{9/2}) \\ 1 & -5/(8\, 2^{3/2}) & 145/(128\, 2^3) & -10765/(3072\, 2^{9/2}) \end{pmatrix}$$

The RHS of (12), $R_2(k)$ say, can be computed as follows. First, we define a matrix $A[1..\infty, 1..\infty]$ by

$$A[1,j] = (d_j - c_j) \left[\frac{3}{2^{3/2}}\right]^j (-1)^j, j = 1..\infty,$$

$$A[i+1,j] = \sum_{l=1}^{j-i} A[i, j-l] A[1,l], i \geq 1, j = 1..\infty.$$

This gives

$$A = \begin{pmatrix} 1/(2\, 2^{3/2}) & -35/(482\, 2^3) & 5005/(2304\, 2^{9/2}) & -1616615/(165888\, 2^6) \\ 0 & 1/(4\, 2^3) & -35/(48\, 2^{9/2}) & 3115/(1152\, 2^6) \\ 0 & 0 & 1/(8\, 2^{9/2}) & -35/(64\, 2^6) \\ 0 & 0 & 0 & 1/(16\, 2^6) \end{pmatrix}$$

and, finally, $R_2(k)$ becomes

$$R_2(k) = \sum_{i=1}^{k} \frac{(-\sqrt{2})^i}{i!} z^{-i} \sum_{l=i}^{k} C[k-i, k-l] A[i, l].$$

It remains to compute the matrix $B[1..\infty, 1..\infty]$:

$$B[k, i] = \sum_{l=i}^{k} C[k-i, k-l] A[i, l]. \tag{16}$$

This leads to

$$B = \begin{pmatrix} 1/(2\,2^{3/2}) & 0 & 0 & 0 \\ -5/(6\,2^3) & 1/(4\,2^3) & 0 & 0 \\ 725/(256\,2^{9/2}) & -25/(32\,2^9) & 1/(8\,2^{9/2}) & 0 \\ -141845/(10368\,2^6) & 1835/(576\,2^6) & -55/(96\,2^6) & 1/(16\,2^6) \end{pmatrix}$$

and $R_2(k)$ is finally given by

$$R_2(k) = \sum_{i=1}^{k} \frac{(-\sqrt{2})^i}{i!} z^{-i} B[k, i].$$

The equivalence $L_2(k) = R_2(k)$ gives the sought for recurrence for Z_k in terms of $Z_1 \ldots Z_{k-1}$.

4 Some properties of the Generalized Airy Distribution

In this section, we seek an asymptotic form for the density $f(x, b)$ of W_b for $x \downarrow 0$. We also recover the asymptotic Gaussian property of the density as $b \uparrow \infty$, already mentioned in Janson [9].

4.1 Preliminary formulae

Setting $b = v/\sqrt{2^{\cdot}\sigma}, u = 2^{\cdot}\alpha$, (4) leads to

$$\frac{1}{\sqrt{2\pi}} v \int_0^\infty e^{-u\sigma} \mathbb{E}\left[e^{-\sqrt{2}\sigma^{3/2} W_{v/\sqrt{2^{\cdot}\sigma}}}\right] e^{-v^2/(2^{4/3}\sigma)} \frac{2^{\cdot} d\sigma}{\sqrt{2}\sigma^{3/2}} = e^{v 2^{\cdot} Ai'(u)/Ai(u)}.$$

Differentiating w.r.t u, this gives

$$\frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-u\sigma} \mathbb{E}\left[e^{-\sqrt{2}\sigma^{3/2} W_{v/\sqrt{2^{\cdot}\sigma}}}\right] e^{-v^2/(2^{4/3}\sigma)} \frac{d\sigma}{\sqrt{2\sigma}} = -e^{v 2^{\cdot} Ai'(u)/Ai(u)} [Ai'(u)/Ai(u)]'.$$

Note that $(Ai'/Ai)' = u - (Ai'/Ai)^2$.
We derive

$$\mathbb{E}\left[e^{-\sqrt{2}\sigma^{3/2} W_{v/\sqrt{2^{\cdot}\sigma}}}\right] e^{-v^2/(2^{4/3}\sigma)} / \sqrt{4\pi\sigma} = \frac{-1}{2\pi i} \int_{-i\infty}^{i\infty} e^{v 2^{\cdot} Ai'(u)/Ai(u)} [Ai'(u)/Ai(u)]' e^{u\sigma} du. \tag{17}$$

Set now $v = b\sqrt{2^{\cdot}\sigma}, z = \sqrt{2}\sigma^{3/2}, \Theta(z, b) = \mathbb{E}[e^{-zW_b}]$. Equ.(17) becomes

$$\frac{2^{1/6}\Theta(z, b) e^{-b^2/2}}{2\sqrt{\pi}} = \frac{-z^{1/3}}{2\pi i} \int_{-i\infty}^{i\infty} e^{b z^{1/3} 2^{\cdot} Ai^p r(u)/Ai(u)} [Ai'(u)/Ai(u)]' e^{u z^{2/3}/2^{\cdot}} du. \tag{18}$$

which proves Thm 1.2. But we know that

$$f(x, b) = \frac{1}{2\pi i} \Re \int_{c-i\infty}^{c+i\infty} e^{xz} \Theta(z, b) dz, \quad c > 0. \tag{19}$$

This justifies the denomination "Generalized Airy Distribution" we have proposed in the introduction. Note that, letting $b \downarrow 0$ in (18), we recover of course the corresponding formula for the Brownian excursion area (i.e. the Airy distribution, see [11]).

4.2 Asymptotics of $f(x,b)$ as $b \uparrow \infty$

From $\psi_1(b)$ and $\psi_2(b)$, we see (as mentioned in [9]) that, as $b \uparrow \infty$, $\mathbb{E}[W_b] \sim \frac{1}{2b}$ and $\mathrm{VAR}[W_b] \sim \frac{1}{4b^4}$. So we consider

$$\Theta(2b^2 z, b)e^{bz} = -\frac{e^{F_1}}{2^{1/6}\sqrt{\pi}i}(2b^2 z)^{1/3}\int_{-i\infty}^{i\infty} e^{F_2}[Ai'(u)/Ai(u)]'du$$

where

$$F_1 := bz + b^2/2, \quad F_2 := b(2b^2 z)^{1/3}2^{\cdot}Ai'(u)/Ai(u) + u(2b^2 z)^{2/3}/2^{\cdot}.$$

We know that

$$\begin{aligned} Ai'(u)/Ai(u) \quad &\sim \quad -\sqrt{u} - \frac{1}{4u} + \frac{5}{32u^{5/2}} + \mathcal{O}(\frac{1}{u^4}), \quad |u| \uparrow \infty, |arg(u)| < \pi, \\ [Ai'(u)/Ai(u)]' \quad &\sim \quad -\frac{1}{2\sqrt{u}} + \mathcal{O}(\frac{1}{u^2}), \quad |u| \uparrow \infty, |arg(u)| < \pi. \end{aligned} \tag{20}$$

We must equilibrate the two terms of F_2 as $b \uparrow \infty$, so we set $u = b^{2/3}w$. This leads to

$$-\frac{e^{F_1}}{2^{1/6}\sqrt{\pi}i}(2b^2 z)^{1/3}b^{2/3}\int_{-i\infty}^{i\infty} e^{F_3}[Ai'(b^{2/3}w)/Ai(b^{2/3}w)]'dw \tag{21}$$

where

$$F_3 \sim (-2^{2/3}z^{1/3}\sqrt{w} + w2^{1/3}z^{2/3})b^2 - \frac{2^{2/3}z^{1/3}b}{4w} + \frac{5z^{1/3}2^{2/3}}{32w^{5/2}} + \mathcal{O}(\frac{1}{b}), \quad b \uparrow \infty.$$

We will now apply the powerful Saddle point method (see, for instance, Flajolet and Sedgewick [6]). We look for w^* such that $F_4'(w^*) = 0$, with $F_4(w) := F_3(w)/b^2$. This gives

$$w^* \sim \frac{2^{2/3}}{4z^{2/3}}\left(1 - \frac{4z}{b} + \frac{5z^2}{b^2} + \mathcal{O}(\frac{1}{b^3})\right)$$

and

$$F_3(w^*) \sim \frac{-b^2}{2} - bz + \frac{z^2}{2} + \mathcal{O}(\frac{1}{b}).$$

Now we set $w = w^* + y$ in F_3. We derive

$$F_1 + F_3 \sim \frac{z^2}{2} + \frac{2^{2/3}z^{4/3}b^2y^2}{2} + \mathcal{O}(\frac{1}{b}).$$

So we finally set $y = i\tau/\sqrt{2^{2/3}z^{4/3}b^2}$ and integrating (21) on τ, we obtain the limiting expression $e^{z^2/2}$ for $\Theta(z,b)$. Setting now $z = i\tilde{z}$, we recover the classical Fourier transform $e^{-\tilde{z}^2/2}$ of the Gaussian distribution.

4.3 Asymptotics of $f(x,b)$ as $x \downarrow 0$. Preliminary analysis

We know from (19) that

$$f(x,b) = \frac{1}{2\pi i}\Re\int_{c-i\infty}^{c+i\infty} e^{xz}\Theta(z,b)dz, \quad c > 0.$$

The limiting distribution as $x \downarrow 0$ is of course related to large $|z|$. Actually we will use a saddle point $z^* = \mathcal{O}(1/x^3)$ as we will see later on. So we turn to (18) and look for an asymptotic equivalent as $|z| \uparrow \infty$. Again we will use the Saddle point method. We note that, in (18) , we have the same poles in $Ai'(u)/Ai(u)$ and $[Ai'(u)/Ai(u)]'$. The zeros a_k of $Ai(u)$ are simple and lie on the negative real u-axis. Let us consider the dominant pole $a_1 = -\alpha_1$ say ($\alpha_1 > 0$): $\alpha_1 = 2.338107410\ldots$ In the neighbourhood of a_1 we have

$$Ai'(u)/Ai(u) \quad \sim \quad \frac{1}{u+\alpha_1} - \frac{1}{3}\alpha_1(u+\alpha_1) + \frac{1}{4}(u+\alpha_1)^2 - \frac{1}{45}\alpha_1^2(u+\alpha_1)^3 + \mathcal{O}((u+\alpha_1)^4), \tag{22}$$

$$[Ai'(u)/Ai(u)]' \quad \sim \quad \frac{-1}{(u+\alpha_1)^2} - \frac{1}{3}\alpha_1 + \frac{1}{2}(u+\alpha_1) + \mathcal{O}((u+\alpha_1)^2). \tag{23}$$

So clearly, we must consider two contributions: the first one, F_5, is related to $\frac{-1}{(u+\alpha_1)^2}$, the other one, F_6, is related to the remaining terms in (23), and $\Theta = F_5 + F_6$.

4.3.1 A Saddle point strategy

In order to build a suitable strategy, let us consider a simpler problem: what is the asymptotic equivalent of

$$F_7(z) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{1/y} e^{yz}\frac{dy}{y^2}, \quad |z| \uparrow \infty, \tag{24}$$

where we have an essential singularity at 0. This asymptotic is of course well known, indeed

$$\int_0^{\infty} \sqrt{x} I_1(2\sqrt{x}) e^{-xv} dv = e^{1/v}/v^2.$$

Hence, with the classical asymptotic equivalent for I_1, for large argument,

$$F_7(z) \sim \frac{1}{\sqrt{\pi}} e^{2\sqrt{z}} \left[\frac{z^{1/4}}{2} - \frac{3}{32 z^{1/4}} - \frac{15}{1024 z^{3/4}} + \mathcal{O}(\frac{1}{z})\right]. \tag{25}$$

Now we set $y = 1/w$ in (24). This leads to

$$F_7 = \frac{i}{2\pi}\int_{C_1} e^{F_8} dw, \tag{26}$$

where $F_8 := w + z/w$ and C_1 is the path image of the contour of (24). (To avoid the origin in (26), we just limit the contour of (24) and close it to the left). To equilibrate the two terms of F_8, we set $w = \sqrt{z}\tau$. This gives

$$F_7 = \frac{i\sqrt{z}}{2\pi}\int_{C_2} e^{\sqrt{z}F_9} d\tau$$

with

$$F_9 = \tau + 1/\tau. \tag{27}$$

The Saddle point of (27) is $\tau^* = 1$ and $F_9(1) = 2$. Now we set $\tau = 1 - s$ in $F_{10} := F_9(1) - F_9(\tau)$. We obtain

$$F_{10} = 1 + s - \frac{1}{1-s}.$$

Set now $F_{10} = v^2$ which gives $v^2 = -s^2[\sum_0^\infty s^k]$. It follows, by Lagrange theorem, that sufficiently near 0, s possess an expansion $s = \sum_1^\infty b_n v^n$. We get

$$s(v) \sim iv + \frac{1}{2}v^2 - \frac{1}{8}iv^3 - \frac{1}{128}iv^5 + \mathcal{O}(v^6)$$

and finally

$$F_7 \sim \frac{i\sqrt{z}}{2\pi} e^{2\sqrt{z}} \int_{-\infty}^{\infty} e^{-\sqrt{z}v^2}\left[\frac{-ds}{dv}\right] dv$$

which, after all algebra, leads indeed to (25).

4.3.2 Asymptotics of $\Theta(z, b)$ as $|z| \uparrow \infty$

Starting from (18), we analyze first

$$F_5 = \frac{z^{1/3}}{\sqrt{\pi} i 2^{1/6}} e^{b^2/2} e^{-\alpha_1 z^{2/3}/2^{\cdot}} \int_{\alpha_1 - i\infty}^{\alpha_1 + i\infty} e^{F_{11}} \frac{dw}{w^2}$$

and

$$F_{11} = b z^{1/3} 2^{\cdot} Ai'(w - \alpha_1)/Ai(w - \alpha_1) + w z^{2/3}/2^{\cdot}.$$

Again, to equilibrate the two terms of F_{11}, we set $w = 2^{\cdot}b/(z^{1/6}\tau)$ (which makes the dominant term of F_{11}, in the neighbourhood of $w = 0$, look like (24)). First, we must carefully analyze the different domains for w and τ. The pole a_1 corresponds to $w = 0$, the next pole a_2 corresponds to $w_2 = a_2 + \alpha_1 = -1.749842034\ldots$, $w = 0^+$ corresponds to $\tau = \infty$, $w = 0^-$ corresponds to $\tau = -\infty$, $w = w_2$ corresponds to $\tau_2 = 2^{\cdot}b/(z^{1/6}w_2)$, all other poles $a_i, i = 3\cdots\infty$ correspond to the range $\tau \in (\tau_2\cdots 0)$, and $\tau = 0$ is an accumulation point of poles.

- The asymptotics (23) are valid for $|w| = o(1)$, i.e. $|\tau| = \Omega(2^{\cdot}b/z^{1/6})$.
- For $\tau = 2^{\cdot}b/z^{1/6}$, we derive $F_{11} = bz^{1/3}2^{\cdot}Ai'(1-\alpha_1)/Ai(1-\alpha_1) + z^{2/3}/2^{\cdot}$.
- For $\tau = \varepsilon e^{I\eta}/z^{1/6}, 0 \le \eta \le \pi/2, \quad \varepsilon = o(1)$, we can use the equivalent(20). This leads to the equivalent

$$\begin{aligned}\Re(F_{11}) \quad &\sim \quad \frac{b^2 z^{2/3}}{\varepsilon}\cos(\eta) - \frac{b^{3/2}z^{1/3}\sqrt{2}}{2\sqrt{\varepsilon}}(2+2\cos(\eta))^{1/2} \\ &+ \quad z^{1/3}2^{1/6}\alpha_1\sqrt{b}\sqrt{\varepsilon}(2+2\cos(\eta))^{1/2}/4 - z^{1/3}\varepsilon\cos(\eta)/4 + \mathcal{O}(\varepsilon^{3/2}).\end{aligned}$$

  The first term corresponds to the w term in F_{11}.

- For $\tau = \varepsilon e^{I(\pi-\eta)}/z^{1/6}, 0 < \eta < \pi/2, \varepsilon = o(1)$, we obtain

$$\begin{aligned}\Re(F_{11}) \quad &\sim \quad -\frac{b^2 z^{2/3}}{\varepsilon}\cos(\eta) - \frac{b^{3/2}z^{1/3}\sqrt{2}}{2\sqrt{\varepsilon}}(2-2\cos(\eta))^{1/2} \\ &+ \quad z^{1/3}2^{1/6}\alpha_1\sqrt{b}\sqrt{\varepsilon}(2-2\cos(\eta))^{1/2}/4 + z^{1/3}\varepsilon\cos(\eta)/4 + \mathcal{O}(\varepsilon^{3/2}).\end{aligned}$$

  For $\eta = o(1)$ this gives

$$\Re(F_{11}) \sim -\frac{b^2 z^{2/3}}{\varepsilon} - \frac{b^{3/2}z^{1/3}\sqrt{2}}{2\sqrt{\varepsilon}}\eta + z^{1/3}2^{1/6}\alpha_1\sqrt{b}\sqrt{\varepsilon}\eta/4 + z^{1/3}\varepsilon/4 + \mathcal{O}(\varepsilon^{3/2}).$$

  Again, the first term corresponds to the w term in F_{11}.

Let us now turn to the Saddle point analysis. It is convenient to set $z = \kappa^6$. F_{11} becomes, in the neighbourhood of $\tau = 1$,

$$F_{11} = \kappa^3 F_{12}$$

with

$$F_{12} \sim \tau + \frac{b}{\tau} - \frac{b^2 2^{2/3}\alpha_1}{3\tau\kappa^2} + \frac{b^3}{2\tau^2\kappa^3} - \frac{2b^4 2^{2/3}\alpha_1^2}{45\tau^3\kappa^4} + \mathcal{O}(\frac{1}{\kappa^5}).$$

We look for a Saddle point τ^* of F_{12}. this is given by $F_{12}'(\tau^*) = 0$, i.e.

$$\tau^* \sim \sqrt{b} - \frac{b^{3/2}2^{2/3}\alpha_1}{6\kappa^2} + \frac{b^2}{2\kappa^3} - \frac{17b^{5/2}2^{1/3}\alpha_1^2}{180\kappa^4} + \frac{b^3 2^{2/3}\alpha_1}{6\kappa^5} + \mathcal{O}(\frac{1}{\kappa^6}). \tag{28}$$

This leads to

$$\mu_1 := F_{12}(\tau^*) \sim 2\sqrt{b} - \frac{b^{3/2}2^{2/3}\alpha_1}{3\kappa^2} + \frac{b^2}{2\kappa^3} - \frac{b^{5/2}2^{1/3}\alpha_1^2}{10\kappa^4} + \mathcal{O}(\frac{1}{\kappa^5}). \tag{29}$$

Now set $\tau = \tau^* - s$ in $F_{13} := \mu_1 - F_{12}(\tau)$. We obtain

$$F_{13} \sim \sqrt{b} + s - \frac{b}{\sqrt{b}-s} - \frac{2^{2/3}\alpha_1 b^{3/2}s^2}{6(\sqrt{b}-s)^2\kappa^2} + \mathcal{O}(\frac{1}{\kappa^4}).$$

We set now $F_{13} = v^2$. This gives

$$s(v) \sim \frac{[-(36\kappa^2 + 6b2^{2/3}\alpha_1)b^{1/2}]^{1/2}\kappa}{6\kappa^2 + b2^{2/3}\alpha_1}v + \frac{3\kappa^2(3\kappa^2 + b2^{2/3}\alpha_1)}{18\kappa^4 + 6\kappa^2 b2^{2/3}\alpha_1 + b^2 2^{1/3}\alpha_1^2}v^2 + \delta_3 v^3 + \mathcal{O}(v^4)$$

and δ_3 is a complicated expression that we will not explicit here. Now we can compute the first terms in the asymptotic expansion of $s(v)$ into κ, insert into F_5 and finally expand the integrand of F_5 into v. Keeping only the constant and v^2 terms (the v term leads to 0), the integration finally leads to

$$F_5 \sim e^{-\alpha_1\kappa^4/2^{\cdot}} e^{\kappa^3\mu_1}\left[\frac{2^{1/2}\kappa^{3/2}}{2b^{3/4}} - \frac{2^{1/6}b^{1/4}\alpha_1}{12\kappa^{1/2}} + \mathcal{O}(\frac{1}{\kappa^{3/2}})\right], z = \kappa^6,$$

where μ_1 is given by (29).

The second contribution F_6 , related to the non-singular part of (23), is similarly computed. This is easier: we only use the transformation $w = 2^{\cdot}b\tau/z^{1/6}$ (there is a singularity only in the exponent part of F_6). Of course μ_1 is identical to (29), and finally we obtain

$$\Theta = F_5 + F_6 = e^{\kappa^3\mu_1}e^{-\alpha_1\kappa^4/2^{\cdot}}F_{14}$$

with

$$F_{14} \sim \frac{2^{1/2}\kappa^{3/2}}{2b^{3/4}} + \frac{b^{1/4}2^{1/6}\alpha_1}{4\kappa^{1/2}} + \mathcal{O}(\frac{1}{\kappa^{3/2}}).$$

4.3.3 Asymptotics of $f(x, b)$ as $x \downarrow 0$

Equ.(19) leads to

$$f(x,b) = \frac{1}{2\pi i}\Re\int_{c-i\infty}^{c+i\infty} F_{14}e^{F_{15}}dz \tag{30}$$

with

$$F_{15} = \kappa^6 x + \kappa^3\mu_1 - \alpha_1\kappa^4/2^{\cdot}, \quad z = \kappa^6.$$

It is convenient to put $x = 1/\eta^2$ and we must consider $\eta \uparrow \infty$. We first set $\kappa = \sqrt{\tau}\eta$ in (30). This leads to

$$\frac{1}{2\pi i}\Re\int_{C_3} F_{14}3\tau^2\eta^6 e^{F_{16}}d\tau$$

where

$$F_{16} \sim \eta^4\left[-\frac{1}{2}\alpha_1\tau^2 2^{2/3} + \tau^3 + \frac{2\tau^{3/2}b^{1/2}}{\eta} - \frac{\tau^{1/2}b^{3/2}2^{2/3}\alpha_1}{3\eta^3} + \frac{b^2}{2\eta^4} - \frac{b^{5/2}2^{1/3}\alpha_1^2}{10\tau^{1/2}\eta^5} + \mathcal{O}(\frac{1}{\eta^6})\right].$$

The Saddle point of $F_{17} := F_{16}/\eta^4$ is given by

$$\tau^* \sim 2^{2/3}\left[\frac{1}{3}\alpha_1 - \frac{3^{1/2}b^{1/2}}{2\alpha_1^{1/2}\eta} - \frac{9b}{8\alpha_1^2\eta^2} + \frac{3^{1/2}b^{2/3}(-405+16\alpha_1^3)}{192\alpha_1^{3/2}\eta^3} + \frac{3b^2(-81+4\alpha_1^3)}{16\alpha_1^5\eta^4} + \mathcal{O}(\frac{1}{\eta^5})\right].$$

As announced, the dominant term of τ^* corresponds to $z^* = \left(\frac{\alpha_1 2^{2/3}}{3}\right)^3\eta^6 \gg 0$.
We obtain

$$\begin{aligned}\mu_2 &= F_{17}(\tau^*) \sim -\frac{2\alpha_1^3}{27} + \frac{4\alpha_1^{3/2}3^{1/2}b^{1/2}}{9\eta} - \frac{3b}{2\eta^2} - \frac{b^{3/2}3^{1/2}(27+16\alpha_1^2)}{72\alpha_1^{3/2}\eta^3} \\ &+ \frac{b^2(-27+16\alpha_1^3)}{16\alpha_1^3\eta^4} - \frac{b^{5/2}(8505-1440\alpha_1^3+256\alpha_1^6)3^{1/2}}{2560\alpha_1^{9/2}\eta^5} + \mathcal{O}(\frac{1}{\eta^6}).\end{aligned} \tag{31}$$

We must indeed compute μ_2 up to the $1/\eta^5$ term as we want a final expression correct up to the $1/\eta$ term. Setting $\tau = \tau^* + s$ in $F_{18} := \mu_2 - F_{17}(\tau)$, putting $F_{18} = v^2$, expanding $s(v)$ into v and performing the integration (we omit the details, this is now routine) we get

$$f(x,b) \sim e^{\mu_2/x^2}\frac{\sqrt{2}}{\sqrt{\pi}}\left[\frac{3^{1/4}\alpha_1^{9/4}}{9x^{11/4}b^{3/4}} - \frac{3^{3/4}\alpha_1^{3/4}}{3x^{9/4}b^{1/4}} + \frac{b^{1/4}3^{1/4}(27+16\alpha_1^3)}{x^{7/4}\alpha_1^{3/4}} + \mathcal{O}(\frac{1}{x^{5/4}})\right]$$

which concludes our (long) derivation of Thm 1.3. Note that the dominant term of μ_2 is exactly the same as the term in the corresponding asymptotic equivalent of the Airy density as $x \downarrow 0$. Note also that the contribution of the other poles is negligible wrt α_1.

4.3.4 Asymptotics of $f(x,b)$ as $x \downarrow 0$, for small b

Returning to (28), we see that, for small b, τ^* approaches 0, which is an accumulation point of poles! So we cannot simply let $b \downarrow 0$ in Thm 1.3. But we can start from Thm 1.2 and expand wrt b:

$$\Theta(z,b) = e^{b^2/2}\Theta_1(z,b),$$

with

$$\Theta_1(z,b) = \Theta_{1,0}(z) + b\Theta_{1,1}(z) + \mathcal{O}(b^2),$$

and

$$\begin{aligned}
\Theta_{1,0}(z) &= \frac{-z^{1/3}}{i\,2^{1/6}\sqrt{\pi}}\int_{-i\infty}^{i\infty}[Ai'(u)/Ai(u)]'e^{uz^{2/3}/2^{\cdot}}du, \\
\Theta_{1,1}(z) &= \frac{-z^{1/3}}{i\,2^{1/6}\sqrt{\pi}}\int_{-i\infty}^{i\infty}z^{1/3}2^{\cdot}Ai'(u)/Ai(u)[Ai'(u)/Ai(u)]'e^{uz^{2/3}/2^{\cdot}}du.
\end{aligned} \tag{32}$$

$\Theta_{1,0}(z)$ is exactly the function which was analyzed in [11]. Let us briefly summarize the basic techniques: from (23), we derive, by classical residues analysis,

$$\Theta_{1,0}(z) = \sum_{k=1}^{\infty}\sqrt{2\pi}ze^{-\alpha_k z^{2/3}/2^{\cdot}}.$$

Equ (19) leads to a dominant term (the contributions of $\alpha_k, k \geq 2$ are exponentially negligible):

$$f_0(x) \sim e^{b^2/2}\frac{1}{2\pi i}\Re\int_{c-i\infty}^{c+i\infty}e^{xz}\sqrt{2\pi}ze^{-\alpha_1 z^{2/3}/2^{\cdot}}dz. \tag{33}$$

and we must find the Saddle point of

$$F_{19} := xz - \alpha_1 z^{2/3}/2^{\cdot}.$$

Set $x = 1/\sqrt{\eta}, \eta \uparrow \infty, z = \tau^3\eta^{3/2}$. This leads to

$$F_{19} = \eta\tau^2(2\tau - \alpha_1 2^{2/3})/2.$$

The Saddle point is given by $\tau^* = 2^{2/3}\alpha_1/3$ and

$$\mu := F_{19}(\tau^*) = -2\alpha_1^3/27.$$

Now we set $\tau = \tau^* + s$ in $F_{20} := \mu - F_{19}(\tau)$, this gives

$$F_{20} = -\alpha_1 2^{2/3}s^2/2 - s^3.$$

Next we put $F_{20} = v^2$. This leads to

$$s = I2^{1/6}v/\alpha_1^{1/2} + 2^{2/3}v^2/(2\alpha_1^2) - 5I2^{1/6}v^3/(4\alpha 1^{7/2}) - 2^{5/3}v^4/\alpha_1^5 + \mathcal{O}(v^5).$$

It is now easy (for MAPLE) to perform the integration in (33). This gives

$$f_0(x) \sim e^{b^2/2}e^{-2\alpha_1^3/(27x^2)}[8\alpha_1^{9/2}/(81x^5) - 35\alpha_1^{3/2}/(27x^3) + \mathcal{O}(1/x)]. \tag{34}$$

From (32) and (22), we derive similarly

$$\Theta_{1,1}(z) = \sum_{k=1}^{\infty}\sqrt{2\pi}z^2/2e^{-\alpha_k z^{2/3}/2^{\cdot}},$$

and this leads to

$$f_1(x) \sim e^{b^2/2}e^{-2\alpha_1^3/(27x^2)}[16\alpha_1^{15/2}/(2187x^8) - 250\alpha_1^{9/2}/(729x^6) + \mathcal{O}(1/x^4)]. \tag{35}$$

which completes the proof of Thm 1.4

5 Conclusion

Using some characteristics of the Airy functions, we have derived some properties of the generalized Airy distribution i.e. the reflected Brownian Bridge area W_b conditioned on its local time b at the origin. Some open problems remain to be solved:

- an asymptotic form for the density $f(x, b)$ as $x \uparrow \infty$. This is not even known for the classical Airy density.
- an explicit form for the density $f(x, b)$.
- an analysis of the local time $t^+(t, a)$ of $B^+(t)$ at a, conditioned on its local time b at the origin. In particular, the moments of $t^+(t, a)$ are the object of work in progress, with B.Gittenberger, in connection with random forests.
- some analytic variations on the generalized Airy distribution (see [4] for the classical Airy distribution)

Let us mention that there are other connections between the hashing with linear probing and the BM: for instance [3] analyses the phase transition for parking blocks, and some coalescence phenomena.

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References

[1] M. Abramowitz and I.A. Stegun. *Handbook of mathematical Functions.* Dover, 1965.

[2] P. Chassaing and S. Janson. A Vervaat-like path transformation for the reflected brownian bridge conditioned on its local time at 0. *Annals of Probability.* To appear.

[3] P. Chassaing and G. Louchard. Phase transition for parking blocks, brownian motion excursion and coalescence. *Random Structures and Algorithms.* To appear.

[4] P. Flajolet and G. Louchard. Analytic variations on the Airy distribution. *Algorithmica*, 31(3):361–377, 2001.

[5] P. Flajolet, P. Poblete, and A. Viola. On the analysis of linear probing hashing. *Algorithmica*, 22:490–515, 1998.

[6] P. Flajolet and R. Sedgewick. The average case analysis of algorithms: Saddle point asymptotics. *INRIA T.R.*, 2376, 1994.

[7] B. Gittenberger and G. Louchard. On the local time density of the reflecting brownian bridge. *Journal of Applied Mathematics and Stochastic Analysis*, 13(2):125–136, 2000.

[8] K. Ito and H.P. McKean, Jr. *Diffusion Processes and their Sample Paths.* Springer, 1974.

[9] S. Janson. Asymptotic distributions for the cost of linear probing hashing. *Random Structures and Algorithms*, 19(3,4):438–471, 2001.

[10] F.B. Knight. The moments of the area under reflected brownian bridge conditioned on its local time at zero. *Journal of Applied Mathematics and Stochastic Analysis*, 13(2):99–124, 2000.

[11] G. Louchard. The brownian excursion area: a numerical analysis. *Computers and Mathematics with Applications*, 10(6):413–417, 1984.

[12] G. Louchard. Kac's formula, Levy's local time and brownian excursion. *Journal of Applied Probability*, 21:479–499, 1984.