The Complete Solution of the Competitive Rank Selection Problem

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Abstract

Two decision makers A and B observe sequentially a given permutation of n uniquely rankable options. A and B have one choice each (without recall) and both must make a choice. At each step only the relative ranks are known, and A has the priority of choice. At the end the (absolute) ranks are compared and the winner is the one who has chosen the better rank.

Extending results by Enns and Ferenstein (1985) [6] and Berry et al (unpublished) [1] this article gives for both A and B the optimal strategy and the corresponding winning probabilities. We show in particular that the limiting winning probabilities for A and B *do* exist, which closes the most important gap in the work of previous authors. This also provides an algorithm for numerically computing the limiting value of these probabilities.

Although our proof is analytic in the strong sense it is interesting to see that it would have been very hard to assemble it without the help of computer algebra. The reason is that the functions we have to investigate display subranges of indices which contrast considerably with respect to error terms when certain terms are replaced by approximations and that computations were very helpful to locate those ranges where a particularly fine tuning of error estimates turned out to be indispensable.

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1 Introduction

Two decision makers A and B observe a random permutation of n uniquely rankable options, denoted by $< 1 >$ (best), $< 2 >, \cdots, < n >$ (worst). Each arrival order (permutation) of these ranks is supposed to be equally probable. Only relative ranks can be observed. A and B have one choice each (without recall) and both must choose an option. At each step A has the priority of choice. The goal of both decision makers is to end up with a better option than the other. We note in particular that A's priority implies that, provided neither A nor B have stopped before, B must choose option number $n-1$. Thus, if we denote by $p(n) A$'s optimal winning probability for n options, one feels that A should have an advantage i.e. that $p(n) > 1/2$ for all n. The central questions of this paper are the form of the optimal strategy as a function of n and the behavior of $p(n)$ as n gets large. We prove that the limiting probability of winning s^* for A does exist, and present a computation algorithm for s^*, leading to $s^* = \cdot 6086026 \cdots$

In a recent work, Chen et al. (1997) [5] studied the same competition problem in the full-information case, where the observations are realizations of n i.i.d. random variables from a known continuous distribution function. This is a conceptually easier problem since with the i.i.d. assumption the optimal behavior of A and B at step k (given neither of them has stopped before) will only depend on $n-k$. The backward equations are therefore easier, but the limiting behavior of $p(n)$ is still intricate. The authors have succeeded to show that in this case $q(n) = 1 - p(n) \uparrow c = 0 \cdot 32756 \cdots$.

The authors also report that Berry et al. (unpublished) [1] studied the rank-based version and proved that B's winning probability for $n \geq 3$ is always less than $1/2 - a$ for some $0 < a \leq 0.1$.

Both Chen et al. [5] as well as Berry et al. [1] were unaware of the work of Enns and Ferenstein [6] (1985) which treats both the no-information problem as well as the full information problem. (The title "The horse game" and the lack of keywords makes it hard for interested authors readers to discover this paper.) See also Enns and Ferenstein ([7], [8]) for further work related with this problem.

Enns and Ferenstein [6] report that the "limit" $s^* := \lim_{n\to\infty} p(n) = 0.6086$ for the no-information problem was obtained empirically. The existence of this limit cannot be taken for granted. (For a somewhat surprising counterexample in a similar situation see for instance Bruss and O'Cinneide (1990) [2]). Indeed, the major effort of the present paper lies in the proof of the existence of s^* .

The paper is organized as follows: Section 2 presents the model, and states the optimization problem. Section 3 describes the optimal strategies for A and B. Section 4 deals with an analysis of approximation on which the proof of convergence is based. Section 5 is devoted to the proof of convergence. Section 6 analyzes the algorithm. Section 7 concludes the paper.

MAPLE was of great help in some complicated computations and graphical presentations, also it showed us where exactly a delicate handling of error terms was needed. The proof itself is purely analytic, however.

2 The Model

Let $r_1, r_2, \cdots, r_n$ be the relative ranks defined by a random permutation $< i_1 >, < i_2 >, \cdots < i_n >$, and $R_1, R_2, \cdots, R_n$ be the corresponding absolute ranks, i.e.

$$r_k = \#\{1 \leq j \leq k : i_j \leq i_k\},$$

and

$$R_k = \#\{1 \leq j \leq n : i_j \leq i_k\}.$$

Further let T be the set of stopping rules adapted to $r_1, r_2, \cdots$, i.e.

$$T = \{\tau : \{\tau = k\} \in \mathcal{F}_k\}$$

where

$$\mathcal{F}_k = \sigma\{r_j : 1 \leq j \leq k\}$$

i.e., the σ-algebra generated by the relative ranks up to step k. If A and B apply the stopping rules $\tau_A, \tau_B \in T$ respectively then, by definition of the game,

$$\{A \text{ wins}\} = \{R_{\tau_A} < R_{\tau_B}\}$$

Therefore A has to choose a stopping rule τ_A maximizing $P(R_{\tau_A} < R_{\tau_B})$ and B to choose a stopping rule minimizing $P(R_{\tau_A} < R_{\tau_B})$. However, we must first show that this is indeed a well-defined optimization problem.

2.1 Stopping rules for A and B

Let T_A denote the set of all stopping rules in T which are available to player A. Note that the priority of choice of A assures that A can stop at any step $k, k = 1, 2, \cdots, n$. Therefore $T_A = T$.

However, for any choice of a stopping rule of A, τ_A say, the set of stopping rules in T available to player B becomes

$$T_B = T_B(\tau_A) = \{\tau \in T : \{\tau_A = k\} \cap \{\tau = k\} = \emptyset\}$$

For each $\tau_A \in T$ the optimal response of B is therefore the choice of a stopping rule $\tau_B^*(\tau_A)$ defined by

$$\tau_B^*(\tau_A) = \arg \sup_{\tau \in T_B(\tau_A)} P(R_\tau < R_{\tau_A})$$

Since $T_B(\tau_A) \subseteq T$ for all $\tau_A \in T$ and T is finite, $\tau_B^*(\tau_A)$ must exist for all $\tau_A \in T$. Therefore A has to choose τ_A in such a way that $\sup P(R_\tau < R_{\tau_A})$ is minimized over all sets of rules $T_B(\tau_A)$, i.e. A must choose the stopping rule

$$\begin{aligned}
\tau_A^* &= \arg\{\inf_{\tau_A \in T} \sup_{\tau \in T_B(\tau_A)} P(R_\tau < R_{\tau_A})\} \\
&= \arg \inf_{\tau_A \in T} P(R_{\tau_B^*(\tau_A)} < R_{\tau_A}) \\
&= \arg \sup_{\tau_A \in T} P(R_{\tau_A} < R_{\tau_B^*(\tau_A)}) \\
&= \arg \sup_{\tau \in T} P(R_\tau < R_{\tau_B^*(\tau)})
\end{aligned}$$

Finally, since $\tau_B^*(\tau)$ exists for all $\tau \in T, \tau_A^*$ is well-defined, and so $\tau_B^* = \tau_B^*(\tau_A^*)$ is well-defined.

2.2 Minimax optimality and optimality

Suppose that player A announces to player B that he will play optimally in the minimax sense, i.e. that he will use the stopping rule τ_A^*. Then player B must use the stopping rule $\tau_B^*(\tau_A^*) = \tau_B^*$. As this problem is a two-person zero-sum game, these rules achieve the Nash equilibrium (as already stated by Enns and Ferenstein (1985)).

It is interesting to point out, however, that A need not announce his choice of the minimax rule. His choice is compelling and therefore predictable for B. This will follow from Lemma 3.1 which shows that there is one and only one rule for A which qualifies to be optimal in any sense. B must therefore respond optimally to A's optimal rule, and both rules achieve then the Nash equilibrium. In this problem we may therefore simply speak of optimal rules without any further specification.

2.3 Example

In the following section we will derive the optimal strategies for player A and player B, but we give here already an example to make things intuitive.

Let $n = 5$, say. Neither A nor B would accept the first observation, because the first observation is the best with probability 1/5 only. Indeed, as we will see, A would not stop on an observation number $k < [\frac{n}{2}]$, unless B has stopped already whereas B's winning probability is bounded below by 0.25 (for all n) so that a winning probability of 1/5 is unacceptable for both players. At step $k = 2$, A would still not stop by the same rule, but B would now stop if the relative rank of the second observation is 1 (and go on otherwise). The reason is that now a relative best so far is the overall best with probability 2/5 which is higher than B's winning probability under optimal play later on (which is $q(5,3) = 1 - p(5,3)$ obtained from Lemmas 3.2 and 3.3). If B stops at $k = 2$, then A's optimal strategy is simply to wait for a better rank. At step $k = 3$ A would also accept a record value (relative rank 1) but B would already be satisfied with a relative rank 2. At step 4, A also becomes more modest and would stop on a record (of course) but also on a relative rank 2. B has now not much choice left. If A does not stop, B has to stop because both players must make a choice.

The overall winning probability for $A(B)$ under the optimal strategy turns out to be 0.6333 (0.3667).

3 Optimal strategies for A and B.

In this section, we give some essential properties of the optimal strategies.

Lemma 3.1 *Given that neither A nor B have stopped before step k, it is optimal for A to stop at step k if and only if the relative rank r_k satisfies*

$$\binom{k}{r_k} \geq \frac{1}{2}\binom{n}{r_k}$$

Proof Suppose that $\tilde{\tau}_A$ is an optimal rule for player A. Let $p(n,k)$ be the conditional winning probability of A under this rule given that $\tilde{\tau}_A \geq k, \tau_B \geq k$, and let $q(n,k) = 1 - p(n,k), 1 \leq k \leq n-1$. We first prove that $p(n,k) \geq 1/2$ for all $1 \leq k \leq n$.

Indeed, suppose that priority was given to A or B according to the outcome of a flip of a fair coin. Then, by symmetry, $p(n,k) = 1/2$. Since A may renounce to his priority according to the same randomized scheme we have $p(n,k) \geq 1/2$.

Secondly, if A turned down option number k although A would win with probability $p_a(n,k) > \frac{1}{2}$ by accepting it, then this would be suboptimal because then B could choose this option and win with the same probability. This, however, is impossible since $p(n,k)+q(n,k) = 1$ and $p(n,k) \geq p_a(n,k)$.

Further, suppose it would be suboptimal for A to stop if this yields a winning probability of exactly $1/2$. But then, if A refuses, B could stop and win with probability $1/2$. This implies again a contradiction, because then A's winning probability is exactly $1/2$ and so A cannot hope to do better. This means that A should stop on k if and only if

$$P(R_k < R_{\tau_B} | \tilde{\tau}_A = k, \tau_B > k) \geq 1/2.$$

This inequality defines $\tilde{\tau}_A$, and no other rule can be optimal for A.

Therefore $\tilde{\tau}_A = \tau_A^*$ and $\tau_B^* = \tau_B^*(\tau_A^*)$.

Once A stops, (given B has not stopped yet), B has clearly to wait for a better rank. Therefore $R_k < R_{\tau_B}$ happens if and only if the first k options contain all r_k best ranks. Since each permutation has probability $1/n!$, we have (as already shown by Enns and Ferenstein)

$$P(R_k < R_{\tau_B} | \tilde{\tau}_A = k, \tau_B > k) = \binom{k}{r_k} \Big/ \binom{n}{r_k}$$

and the proof is complete. ∎

For the following, let $\mathbf{1}\{\}$ be the indicator function and let

$$b(n,k,r) := \binom{k}{r} \Big/ \binom{n}{r}.$$

Further let $p(n,k|r)$ denote A's optimal winning probability given that $\tau_A^* \geq k$ and $\tau_B^* \geq k$ and that $r_k = r$. Then the $p(n,k|r)$ satisfy the following equality.

Lemma 3.2

$$\begin{aligned} p(n,k|r) &= b(n,k,r)\mathbf{1}\{b(n,k,r) \geq \frac{1}{2}\} \\ &+ (1 - b(n,k,r))\mathbf{1}\{1 - p(n,k+1) \leq b(n,k,r) < \frac{1}{2}\} \\ &+ p(n,k+1)\mathbf{1}\{b(n,k,r) < 1 - p(n,k+1)\} \end{aligned}$$

Proof Let $q(n,k) = 1 - p(n,k)$. By Lemma 3.1 A stops at k if and only if $b(n,k,r) \geq \frac{1}{2}$ and wins with probability $b(n,k,r)$. If A does not stop then it is optimal for B to stop if $b(n,k,r) \geq q(n,k+1)$, in which case A wins if and only if B loses, i.e. with probability $1 - b(n,k,r)$. This explains the second term of the RHS. Finally, if B turns down option number k as well, both players will enter step $k+1$ and thus A's winning probability becomes $p(n,k+1)$. This proves the lemma. ■

The trick is now to link the $p(n,k)$ and $p(n,k+1)$ via Rényi's theorem on relative ranks (Rényi 1962) [11].

Lemma 3.3

$$p(n,k) = \sum_{r=1}^{k} \frac{1}{k} p(n,k|r)$$

Proof It follows immediately from Rényi's theorem that $P(r_k = j|\mathcal{F}_{k-1}) = P(r_k = j) = 1/k, j = 1, 2, \cdots, k$.

Thus

$$p(n,k) = E(p(n,k,r_k)|\mathcal{F}_{k-1}) = \frac{1}{k}\sum_{r=1}^{k} p(n,k|r).$$

■

We note that $p(n,n)$ is not defined, because at step n at least one of the players must have stopped already. Therefore it suffices to define the $p(n,k|r)$ for $1 \leq r \leq k \leq n-1$ only. Step $n-1$ is special in the sense that if A does not stop, B has to stop. Therefore, from Lemma 3.1 and Lemma 3.2,

$$\begin{aligned} p(n,n-1|r) &= b(n,n-1,r)\mathbf{1}\{b(n,n-1,r) \geq \frac{1}{2}\} \\ &+ (1 - b(n,n-1,r))(1 - \mathbf{1}\{b(n,n-1,r) \geq \frac{1}{2}\} \\ &= (1 - \frac{r}{n})\mathbf{1}\{r \leq \frac{n}{2}\} + \frac{r}{n}\mathbf{1}\{r > \frac{n}{2}\}, \end{aligned}$$

and thus from Lemma 3.3,

$$\begin{aligned} p(n,n-1) &= \frac{1}{n-1}(\sum_{r=1}^{[n/2]}(1 - \frac{r}{n}) + \sum_{r=[n/2]+1}^{n-1}\frac{r}{n}) \\ &= (2n(n-1))^{-1}(2(n-1)[\frac{n}{2}] - 2[\frac{n}{2}]^2 + n(n-1)). \end{aligned}$$

Corollary 3.4

$$p(n,n-1) \uparrow 3/4 \qquad as \qquad n \to \infty.$$

Proof It is straightforward to check that the sequences $(p(2n+1,2n))_{n=2,3,\cdots}$ and $(p(2n+2,2n+1))_{n=2,3,\cdots}$ are identical, and that for n even,

$$p(n,n-1) = \frac{3n-4}{4n-4} \uparrow 3/4.$$

■

We first transform lemma 3.2 and 3.3 into a more suitable form. We replace k by $n-1-k$, use the notation $u^{\underline{k}} := \prod_0^{k-1}(u-i)$ (falling factorial) and the abbreviation

$$C(n,k,r) := \frac{(n-1-k)^{\underline{r}}}{n^{\underline{r}}}.$$

More precisely, we set

$$p_n(k) := (p(n,n-k-1).$$

Then we have (see corollary 3.4)

$$p_n(0) = \frac{6\lceil n/2\rceil - 4}{8\lceil n/2\rceil - 4}$$

and by Lemmata 3.2 and 3.3

$$\begin{aligned} p_n(k) &= \sum_{r=1}^{n-1-k} [C(n,k,r)\mathbf{1}\{C(n,k,r) \geq \frac{1}{2}\} \\ &+ (1-C(n,k,r))\mathbf{1}\{1-p_n(k-1) \leq C(n,k,r) < \frac{1}{2}\} \qquad (1) \\ &+ p_n(k-1)\mathbf{1}\{C(n,k,r) < 1-p_n(k-1)\}]/(n-1-k) \\ &= \frac{1}{n-k-1}\left(\sum_{r=1}^{r_1} C(n,k,r) + \sum_{r=r_1+1}^{r_2}(1-C(n,k,r)) + (n-k-1-r_2)p_n(k-1)\right), \end{aligned}$$

where r_1, r_2 are defined by

$$C(n,k,r_1) \geq \frac{1}{2}, \quad C(n,k,r_1+1) < \frac{1}{2}$$

and by

$$C(n,k,r_2) \geq 1-p_n(k-1), \quad C(n,k,r_2+1) \leq 1-p_n(k-1).$$

It is a-priori known that $\frac{1}{2} \leq p_n(k) \leq \frac{3}{4}$ for all n,k (see [3]). Furthermore, $C(n,k,r)$ is strictly decreasing in r. Hence r_1, r_2 are well defined and $r_1 \leq r_2$.

Our aim is to get a precise information about the behaviour of $p_n(n-2) = p(n,1)$ and to show that the limit

$$\lim_{n\to\infty} p_n(n-2)$$

exists. The next section prepares the tools for the proof.

4 Preliminaries

In this section we collect some asymptotic expansions and properties of the solution of a differential equation which will be needed in the proof of the convergence of $p_n(n-2)$.

4.1 Approximations for $C(n,k,r)$

In [3] uniform bounds for $p(n,k) = p_n(n-k-1)$ are proved which are now very useful as a-priori bounds. We recall the main result.

Lemma 4.1 *The $p(n,k)$ and $q(n,k)$ are uniformly bounded, namely $1/4 \leq q(n,k) \leq 1/2 \leq p(n,k) \leq 3/4$ for all $n, k \leq n-1$.*

Lemma 4.1 implies that B would never stop before option number $[\frac{n}{4}]$ under optimal behavior.

Indeed, suppose the contrary, i.e. B would stop at an option number $j < [\frac{n}{4}]$. Since relative ranks in the future are independent of those of the past and all arrival orders are equally likely $< 1 >$ arrives later with probability $1-\frac{j}{n}$ and can thus be chosen by A. (We know that A would never stop before step $[\frac{n}{2}]$ unless B has stopped before). Thus A would win with probability $> 3/4$, which contradicts by Lemma 4.1 the optimal behavior of B.

Now, if neither A nor B would stop before $[\frac{n}{4}]$, all $p(n,k)$ and $q(n,k) = 1 - p(n,k)$ with $k < [\frac{n}{4}]$ must be constant.

The next two lemmata provide estimates for $C(n,k,r)$ which will be needed in the convergence proof. It follows from (1) and the definitions of r_1 and r_2 that we can confine our interest to the ranges given in the following lemma.

Lemma 4.2 *$C(n,k,r)$ can be approximated in the following way:*

$$C(n,k,r) = \begin{cases} \left(1-\frac{r}{n}\right)^{k+1}(1+\mathcal{O}(k/n)) & \text{if } k = \mathcal{O}(\sqrt{n}),\ r \leq n/4, \text{ and } kr = \mathcal{O}(n), \\ \left(1-\frac{k+1}{n}\right)^{r}(1+\mathcal{O}(r/n)) & \text{if } r = \mathcal{O}(\sqrt{n}) \text{ and } kr = \mathcal{O}(n). \end{cases}$$

Proof First, suppose that $k = \mathcal{O}(\sqrt{n})$, $r \leq n/4$, and $kr = \mathcal{O}(n)$. Then

$$\begin{aligned} C(n,k,r) &= \left(1-\frac{k+1}{n}\right)\left(1-\frac{k+1}{n-1}\right)\cdots\left(1-\frac{k+1}{n-r+1}\right) \\ &= \exp\left(-\frac{k+1}{n} - \frac{k+1}{n-1} - \cdots - \frac{k+1}{n-r+1} + \mathcal{O}\left(\frac{rk^2}{n^2}\right)\right) \\ &= \exp\left(-(k+1)\ell n\frac{n}{n-r} + \mathcal{O}\left(\frac{k}{n}\right)\right) \\ &= \left(1-\frac{r}{n}\right)^{k+1}\left(1+\mathcal{O}\left(\frac{k}{n}\right)\right). \end{aligned}$$

Next, suppose that $r = \mathcal{O}(\sqrt{n})$ and $kr = \mathcal{O}(n)$. Here we use the representation

$$C(n,k,r) = \left(1-\frac{k+1}{n}\right)^r \frac{1-\frac{k+1}{n-1}}{1-\frac{k+1}{n}} \cdots \frac{1-\frac{k+1}{n-r+1}}{1-\frac{k+1}{n}}$$

and the estimate

$$\frac{1-\frac{k+1}{n-j}}{1-\frac{k+1}{n}} = e^{\mathcal{O}(kj/n^2)} = e^{\mathcal{O}(kr/n^2)} = e^{\mathcal{O}(1/n)}$$

which imply

$$C(n,k,r) = \left(1-\frac{k+1}{n}\right)^r e^{\mathcal{O}(r/n)}.$$

■

Lemma 4.3 *Let $\overline{r}$ be $\mathcal{O}(n/k)$. Then*

$$\sum_{r=1}^{\overline{r}} C(n,k,r) = \begin{cases} \frac{n}{k+2}\left(1-\left(1-\frac{\overline{r}}{n}\right)^{k+2}\right)+\mathcal{O}(1) & \text{if } k=\mathcal{O}(\sqrt{n}) \text{ and } k\overline{r}=\mathcal{O}(n), \\ \frac{n-k-1}{k+1}\left(1-\left(1-\frac{k+1}{n}\right)^{\overline{r}}\right)+\mathcal{O}(\overline{r}/k) & \text{if } \overline{r}=\mathcal{O}(\sqrt{n}) \text{ and } k\overline{r}=\mathcal{O}(n). \end{cases}$$

Proof First, suppose that $k=\mathcal{O}(\sqrt{n})$ and $k\overline{r}=\mathcal{O}(n)$. Since

$$\begin{aligned} \sum_{r=1}^{\overline{r}}\left(1-\frac{r}{n}\right)^{k+1} &= n\int_0^{\overline{r}/n}(1-x)^{k+1}\,dx+\mathcal{O}(1) \\ &= \frac{n}{k+2}\left(1-\left(1-\frac{\overline{r}}{n}\right)^{k+2}\right)+\mathcal{O}(1) \end{aligned}$$

we immediately obtain the first estimate.

In the second case (where if $k=\mathcal{O}(\sqrt{n})$ and $k\overline{r}=\mathcal{O}(n)$) we have

$$\begin{aligned} \sum_{r=1}^{\overline{r}} C(n,k,r) &= \left(1+\mathcal{O}\left(\frac{\overline{r}}{n}\right)\right)\sum_{r=1}^{\overline{r}}\left(1-\frac{k+1}{n}\right)^r \\ &= \left(1+\mathcal{O}\left(\frac{\overline{r}}{n}\right)\right)\left(1-\frac{k+1}{n}\right)\frac{1-\left(1-\frac{k+1}{n}\right)^{\overline{r}}}{1-\left(1-\frac{k+1}{n}\right)} \\ &= \frac{n-k-1}{k+1}\left(1-\left(1-\frac{k+1}{n}\right)^{\overline{r}}\right)+\mathcal{O}(\overline{r}/k). \end{aligned}$$

This proves the lemma. ■

4.2 A singular differential equation

We consider the following differential equation:

$$p'(s) = \frac{-r_2}{1-s}p(s)+\frac{2}{s}\left(1-(1-s)^{r_1}\right)-\frac{1}{s}\left(1-(1-s)^{r_2}\right)+\frac{r_2-r_1}{1-s}, \tag{2}$$

with the initial condition

$$p(s) = p^* + \mathcal{O}(s) \qquad \text{as } s \to 0+, \tag{3}$$

where $p^* = 0.672437586022483\cdots$ is the solution of

$$\ell n(1-p^*)(p^*-1) + (1-p^*) - \ell n(2) = 0, \tag{4}$$

and r_1, r_2 are non-negative integers (depending on s and $p(s)$) determined by

$$(1-s)^{r_1} \geq \frac{1}{2}, \quad (1-s)^{r_1+1} < \frac{1}{2}$$

, and by

$$(1-s)^{r_2} \geq 1 - p(s), \quad (1-s)^{r_2+1} < 1 - p(s).$$

Our first aim is to show the existence of such a solution $p(s)$. In particular we are interested in $p(3/4)$ which can be evaluated approximately to $p(3/4) = s^* = 0.6086\cdots$ by numerical simulation. (It is obvious that $p(s)$ is constant for $s \geq 3/4$, i.e. $p(s) = p(3/4)$ for $s \geq 3/4$.)

Lemma 4.4 *Let $p(s)$, $s > 0$, denote a solution of (2) and set*

$$q(s) = p(s) - p^*,$$

Then there exist positive constants $s_0, q_0 > 0$ such that $q(s)$, $0 < s \leq s_0 < 1$, is the solution of the differential equation

$$q'(s) = \frac{q(s)}{s} F(q(s)) + G(q(s)) + sH(s, q(s)) \tag{5}$$

where $H(s,q)$, $0 < s \leq s_0$, $|q| \leq q_0$, is a bounded function,

$$G(q) = \frac{1}{2}\left(-(1-p^*) + (p^*+q)\ell n\left(1 - \frac{q}{1-p^*}\right) + q\ \ell n(1-p^*)\right),$$

and

$$F(q) = \ell n(1-p^*) - \frac{1-p^*-q}{q}\ell n\left(1 - \frac{q}{1-p^*}\right) - 1.$$

Proof We can write

$$(1-s)^{r_1} = \frac{1}{2}(1 + sr_1(s))$$

and

$$(1-s)^{r_2} = (1-p^*-q(s))(1+sr_2(s)),$$

where $r_1(s)$ and $r_2(s)$ are bounded functions. Using this representation we have

$$r_1 = \frac{1}{s}\left(\ell n 2 - s\left(\frac{\ell n 2}{2} + r_1(s)\right)\right) + \mathcal{O}(s)$$

and

$$
\begin{aligned}
r_2 &= \frac{1}{s}\Bigg(-\ell n(1-p^*) - \ell n\left(1-\frac{q(s)}{1-p^*}\right) \\
&\qquad + s\left(\frac{\ell n(1-p^*)}{2} + \frac{\ell n\left(1-\frac{q(s)}{1-p^*}\right)}{2} - r_2(s)\right)\Bigg) + \mathcal{O}(s).
\end{aligned}
$$

Hence, we have

$$
\begin{aligned}
q'(s) &= \frac{1}{s}\left(1-p^* + (p^*-1)\ell n(1-p^*) - \ell n 2\right) \\
&+ \frac{1}{s}\left(-q(s) + (p^*+q(s)-1)\ell n\left(1-\frac{q(s)}{1-p^*}\right) + q(s)\ell n(1-p^*)\right) \\
&- r_1(s) - q(s)r_2(s) + (1-p^*)r_2(s) - \frac{1}{2}\ell n(1-p^*) - \frac{1}{2}\ell n\left(1-\frac{q(s)}{1-p^*}\right) \\
&- \frac{1}{2}\ell n 2 - r_2(s) + r_1(s) + \frac{1}{2}p^*\ell n(1-p^*) + \frac{1}{2}p^*\ell n\left(1-\frac{q(s)}{1-p^*}\right) + p^* r_2(s) \\
&+ \frac{1}{2}q(s)\ell n(1-p^*) + \frac{1}{2}q(s)\ell n\left(1-\frac{q(s)}{1-p^*}\right) + q(s)r_2(s) \\
&+ \mathcal{O}(s) \\
&= \frac{q(s)}{s}F(q(s)) + G(q(s)) + sH(s,q(s)).
\end{aligned}
$$

From this representation you see that a solution of the form $q(s) = p(s) - p^* = p's + o(s)$ is expected, where

$$
p' = -\frac{1-p^*}{2(1-\ell n(1-p^*))} = -\frac{1}{2}\frac{(1-p^*)^2}{\ell n 2} = -\cdot 0773985223196752\cdots. \tag{6}
$$

Please note that $r_1(s)$ and $r_2(s)$ do not appear in $F(q(s))$ and $G(q(s))$. Only $H(s,q(s))$ depends on $r_1(s)$ and $r_2(s)$. ■

From Lemma 4.4 we immediately derive the following corollary.

Corollary 4.5 *Let $p(s)$, $s>0$, denote a solution of (2) and set*

$$
u(s) = p(s) - p^* - p's,
$$

Then there exist positive constants $s_0, r_0 > 0$ such that $u(s)$, $0 < s \le s_0$, is the solution of the differential equation

$$
u'(s) = \frac{u(s)}{s}F(u(s)) + u(s)G_1(u(s)) + sH_1(s,u(s)) \tag{7}
$$

where $G_1(r)$ and $H_1(s,r)$, $0 < s \le s_0$, $|r| \le r_0$, are bounded functions.

Now, we can describe the idea, how to construct a solution $p(s)$ with $p(s) = p^* + O(s)$ as $s \to 0+$. Let s_i be an arbitrary sequence of positive numbers such that $s_i \to 0$ monotonically as $i \to \infty$ and consider the functions $p_i(s)$, $s \geq s_i$, as the solution of (2) with initial condition $p_i(s_i) = p^* + p's_i$. For convenience we will use the special sequence $\beta_i > 0$ defined by

$$(1-\beta_i)^i = \frac{1}{2},$$

respectively

$$\beta_i = 1 - 2^{-1/i} = \frac{\ell n 2}{i} + \mathcal{O}(i^{-2}).$$

We will show that

$$p(s) = \lim_{i\to\infty} p_i(s)$$

exists and provide uniform error estimates.

Lemma 4.6 *Let $p_i(s)$, $s \geq \beta_i$ denote the solution of (2) with $p_i(\beta_i) = p^* + p'\beta_i$. Then*

$$p_j(\beta_i) = p_i(\beta_i) + \mathcal{O}(\beta_i^2)$$

uniformly for $j \geq i$ as $i \to \infty$.

Proof Let C be a sufficiently large constant such that

$$|rG_1(r) + sH_1(s,r)| \leq \frac{C}{100}s$$

if $|r| \leq s$ and $0 < s \leq s_0$. Then we have for $Cs^2 \leq |r| \leq s$

$$|rG_1(r) + sH_1(s,r)| \leq \frac{1}{100}\frac{|r|}{s},$$

respectively $rF(r)/s$ dominates the behaviour of the right hand side of (7) if $Cs^2 \leq |r| \leq s$. However, if we consider the solution $w_j(s)$, $s \geq \beta_j$, with $w_j(\beta_j) = 0$ (which corresponds to the solution $p_j(s) = p^* + p's + w_j(s)$ of (2) with $p_j(\beta_j) = p^* + p'\beta_j$) then we start at the point $s = \beta_j$, $r = 0$, which is not contained in the region $Cs^2 \leq |r| \leq s$.

We will now show that we always have $|w_j(s)| \leq Cs^2$, i.e. $w_j(s)$ does not pass the curves $r = \pm Cs^2$. Assuming the contrary there exists $s_1 > \beta_j$ with $w_j(s_1) = Cs_1^2$ and $w_j'(s_1) > 2Cs_1$. (Without loss of generality we assume that $w_j(s_1) > 0$.) However, since $F(0) = \ell n(1-p^*) = -1.11607666503374\cdots$ we have

$$w_j'(s_1) = \frac{w_j(s_1)}{s_1}F(w_j(s_1)) + w_j(s_1)G_1(w_j(s_1)) + s_1H_1(s_1, w_j(s_1)) < 0.$$

Thus $|w_j(s)| \leq Cs^2$ which implies

$$p_j(\beta_i) = p_i(\beta_i) + \mathcal{O}(\beta_i^2)$$

uniformly for $j \geq i$ as $i \to \infty$. ■

Lemma 4.7 *Let $p_i(s)$, $s \ge \beta_i$ denote the solution of (2) with $p_i(\beta_i) = p^* + p'\beta_i$. Then*

$$\max_{\beta_i \le s \le 3/4} |p_j(s) - p_i(s)| = \mathcal{O}(\beta_i^2)$$

uniformly for $j \ge i$ as $i \to \infty$.

Proof The first observation is that the differential equation (2) is piecewisely linear. More precisely, if we define $\tilde\beta_{k,i}$ by

$$(1-\tilde\beta_{k,i})^k = 1 - p_i(\tilde\beta_{k,i})$$

and if we assume that $\tilde\beta_{k,i} \le \beta_{l+1} \le \tilde\beta_{k-1,i}$ then for $s \in [\tilde\beta_{k,i}, \beta_{l+1}]$ we have $r_1 = l+1$ and $r_2 = k-1$ and (2) reads as

$$p_i'(s) = -\frac{k}{1-s} p_i(s) + f_{l,k}(s),$$

where

$$f_{l,k}(s) = \frac{2}{s}\left(1-(1-s)^l\right) - \frac{1}{s}\left(1-(1-s)^k\right) + \frac{k-l}{1-s}.$$

Hence, if we set

$$q(s) = \left(\frac{1-\tilde\beta_{k,i}}{1-s}\right)^k p_i(s)$$

then

$$q'(s) = \left(\frac{1-\tilde\beta_{k,i}}{1-s}\right)^k f_{l,k}(s)$$

and consequently

$$p_i(\beta_{l+1})\left(\frac{1-\tilde\beta_{k,i}}{1-\beta_{l+1}}\right)^k = p_i(\tilde\beta_{k,i}) + \int_{\tilde\beta_{k,i}}^{\beta_{l+1}} \left(\frac{1-\tilde\beta_{k,i}}{1-t}\right)^k f_{l,k}(t)\, dt.$$

Obviously we can do something similar for $p_j(s)$. The difference is that (usually) $\tilde\beta_{k,i} \ne \tilde\beta_{k,j}$.

Let k_0 be defined by $\beta_{i+1} \le \tilde\beta_{k_0,i} < \beta_i$. Then we also have for $j \ge i$ (compare with Lemma 4.6)

$$|p_i(\tilde\beta_{k_0,i}) - p_j(\tilde\beta_{k_0,i})| \le C\beta_i^2.$$

We will now show inductively that

$$\max_{\tilde\beta_{k-1,i} \le s \le \tilde\beta_{k,i}} |p_i(s) - p_j(s)| = \mathcal{O}(\beta_i^2). \tag{8}$$

for all $k \le k_0$. For this purpose we will first assume that k is sufficiently large, i.e. $k \ge \overline{k}$, where $\overline{k}$ is an absolute constant and will be fixed in the sequel. The first step is to show that

$$|p_i(\tilde\beta_{k,i}) - p_j(\tilde\beta_{k,i})| \le C\beta_i^2. \tag{9}$$

for $\overline{k} \le k \le k_0$.

Suppose that (9) is satisfied for some k. Since

$$\beta_l \sim \frac{\ell n 2}{l} \quad \text{and} \quad \tilde{\beta}_{ki} \sim \frac{-\ell n(1-p^*)}{k}$$

it follows that (asymptotically) there is at most one β_{l+1} contained in an interval $[\tilde{\beta}_{k,i}, \tilde{\beta}_{k-1,i}]$. In order to demonstrate our method let us assume that this situation occurs, and (for simplicity) let us assume that

$$\tilde{\beta}_{k,i} < \tilde{\beta}_{k,j} < \beta_{l+1} < \tilde{\beta}_{k-1,i} < \tilde{\beta}_{k-1,j}.$$

(All other cases can be treated in the same way.) Since

$$\begin{aligned} \frac{d}{ds}(1-s)^k \Big|_{s=\tilde{\beta}_{k,i}} &= -k(1-\tilde{\beta}_{k\cdot})^{k-1} \\ &= -k(1-p^* + \mathcal{O}(\tilde{\beta}_{k,i})) \end{aligned}$$

and $p_j(s)$ is monotonically decreasing it follows by an elementary geometric observation that

$$|\tilde{\beta}_{ki} - \tilde{\beta}_{kj}| \le \frac{C+\varepsilon}{1-p^*} \frac{\beta_i^2}{k},$$

where $\varepsilon > 0$ is a sufficiently small number (to be fixed in the sequel) and i, j, k are sufficiently large)

Now we have

$$p_i(\tilde{\beta}_{k,j}) \left(\frac{1-\tilde{\beta}_{k,i}}{1-\tilde{\beta}_{k,j}} \right)^k = p_i(\tilde{\beta}_{k,i}) + \int_{\tilde{\beta}_{k,i}}^{\tilde{\beta}_{k,j}} \left(\frac{1-\tilde{\beta}_{k,i}}{1-t} \right)^k f_{l,k}(t)\, dt$$

and

$$p_j(\tilde{\beta}_{k,j}) \left(\frac{1-\tilde{\beta}_{k,i}}{1-\tilde{\beta}_{k,j}} \right)^{k-1} = p_j(\tilde{\beta}_{k,i}) + \int_{\tilde{\beta}_{k,i}}^{\tilde{\beta}_{k,j}} \left(\frac{1-\tilde{\beta}_{k,i}}{1-t} \right)^{k-1} f_{l,k-1}(t)\, dt.$$

Since

$$|f_{l,k}(s) - f_{l,k-1}(s)| = |(1-p^*) + \mathcal{O}(s)| \le (1-p^*) + \varepsilon,$$

(where $\varepsilon > 0$ is a sufficiently small number and $0 < s \le s_0$ is sufficiently small depending on ε) it is an easy task to show that

$$\left| \int_{\tilde{\beta}_{k,i}}^{\tilde{\beta}_{k,j}} \left(\frac{1-\tilde{\beta}_{k,i}}{1-t} \right)^k f_{l,k}(t)\, dt - \int_{\tilde{\beta}_{k,i}}^{\tilde{\beta}_{k,j}} \left(\frac{1-\tilde{\beta}_{k,i}}{1-t} \right)^{k-1} f_{l,k-1}(t)\, dt \right| \le (C+3\varepsilon) \frac{\beta_i^2}{k}$$

and consequently

$$\left| p_i(\tilde{\beta}_{k,j}) - p_j(\tilde{\beta}_{k,j}) \right| \le C\beta_i^2 + (C+4\varepsilon)\frac{\beta_i^2}{k}.$$

Since

$$p_i(\beta_{l+1}) \left(\frac{1-\tilde{\beta}_{k,j}}{1-\beta_{l+1}} \right)^k = p_i(\tilde{\beta}_{k,j}) + \int_{\tilde{\beta}_{k,j}}^{\beta_{l+1}} \left(\frac{1-\tilde{\beta}_{k,j}}{1-t} \right)^k f_{l,k}(t)\, dt$$

and

$$p_j(\beta_{l+1})\left(\frac{1-\tilde{\beta}_{k,j}}{1-\beta_{l+1}}\right)^k = p_j(\tilde{\beta}_{k,j}) + \int_{\tilde{\beta}_{k,j}}^{\beta_{l+1}} \left(\frac{1-\tilde{\beta}_{k,j}}{1-t}\right)^k f_{l,k}(t)\,dt$$

we have

$$p_i(\beta_{l+1}) - p_j(\beta_{l+1}) = \left(\frac{1-\beta_{l+1}}{1-\tilde{\beta}_{k-1,i}}\right)^k (p_i(\tilde{\beta}_{k-1,i}) - p_j(\tilde{\beta}_{k-1,i}))$$

and

$$p_i(\tilde{\beta}_{k-1,i}) - p_j(\tilde{\beta}_{k-1,i}) = \left(\frac{1-\tilde{\beta}_{k-1,i}}{1-\beta_{l+1}}\right)^k (p_i(\beta_{l+1}) - p_j(\beta_{l+1}))$$

Hence, since $\ell n(1-p^*) = -1.11607\cdots < -1$ we obtain for sufficiently small $\varepsilon > 0$ (and for sufficiently large $k \geq \overline{k}$)

$$\begin{aligned}
\left|p_i(\tilde{\beta}_{k-1,i}) - p_j(\tilde{\beta}_{k-1,i})\right| &\leq \left(\frac{1-\tilde{\beta}_{k-1,i}}{1-\tilde{\beta}_{k,i}}\right)^k \left(C\beta_i^2 + (C+4\varepsilon)\frac{\beta_i^2}{k}\right) \\
&\leq \left(1 + \frac{\ell n(1-p^*)+\varepsilon}{k}\right)\left(C\beta_i^2 + (C+4\varepsilon)\frac{\beta_i^2}{k}\right) \\
&\leq C\beta_i^2.
\end{aligned}$$

This proves (9) for $k \geq \overline{k}$.

It is also evident from the above proof that (9) implies (8) for $k \geq \overline{k}$.

It remains to consider finitely many steps, i.e. the intervals $[\tilde{\beta}_{k,i}, \tilde{\beta}_{k-1,i}]$ for $k < \overline{k}$. We can proceed as above where the constant C has to be (possibly) enlarged but only finitely many times by a finite factor. This proves the lemma. ■

It is now immediate that the limit

$$\lim_{i\to\infty} p_i(s) = p(s)$$

exists and that the convergence is uniform. Note that we also have the expansion

$$p(s) = p^* + p's + \mathcal{O}(s^2) \tag{10}$$

as $s \to 0+$.

5 Proof of Convergence

We distinguish three ranges for k, with different behavior: these are $k \leq \sqrt{n}$, $\sqrt{n} \leq k \leq n^{3/4}$, and $n^{3/4} \leq k \leq \frac{n}{2}$. Let us analyze these three ranges.

5.1 Subrange 1: $k \le n^{1/2}$

In a first step we show that for $k \to \infty$, but relatively small with respect to n (i.e. $k \le \sqrt{n}$), we have $p_n(k) \sim p^* = 0.672437586022483\cdots$ (This in conformity with Enns et al. [7], sec. 3). More precisely, we will prove the following property:

Proposition 5.1 *We have*

$$p_n(k) = p^* - \frac{D_2^*}{(1+D_1)k} + \mathcal{O}(k^{D_1}) \tag{11}$$

uniformly for $k \le \sqrt{n}$ as $k, n \to \infty$, where

$$D_1 = \ell n(1-p^*) = -1 \cdot 11607666503374\cdots$$

and

$$D_2^* = \ell n^2(2)/2 + \ell n^2(1-p^*)\frac{1}{2}(p^*-1) = \cdot 036216193421872\cdots.$$

Proof Since $k \le \sqrt{n}$ we can use the approximation

$$C(n,k,r) = \left(1 - \frac{r}{n}\right)^{k+1}(1 + \mathcal{O}(k/n))$$

for $r = \mathcal{O}(n/k)$ (see Lemma 4.2). Observe that

$$\frac{r_1}{n} = 1 - 2^{-1/(k+1)}(1 + \mathcal{O}(1/n)) = \frac{\ell n 2}{k} + \mathcal{O}(1/k^2)$$

and that

$$\frac{r_2}{n} = 1 - (1 - p_n(k-1))^{1/(k+1)}(1 + \mathcal{O}(1/n)) = \frac{-\ell n(1 - p_n(k-1))}{k} + \mathcal{O}(1/k^2).$$

(Note that, according to Lemma 4.1, $1/2 \le p_n(k-1) \le 3/4$.) Hence, by Lemma 4.3

$$\sum_{r=1}^{r_1} C(n,k,r) = \frac{n}{k+2}\left(1 - \frac{1}{2} + \frac{\ell n 2}{k} + \mathcal{O}(1/k^2)\right) + \mathcal{O}(1) = \frac{n}{2k} + \mathcal{O}(n/k^2)$$

and

$$\begin{aligned}
\sum_{r=r_1+1}^{r_2}(1 - C(n,k,r)) &= r_2 - r_1 - \sum_{r=1}^{r_2} C(n,k,r) + \sum_{r=1}^{r_1} C(n,k,r) \\
&= \frac{n}{k}\left(-\ell n(1 - p_n(k-1) - \ell n 2 - p_n(k-1) + \frac{1}{2}\right) + \mathcal{O}(n/k^2)
\end{aligned}$$

and we can rewrite recurrence (1) in the form

$$\begin{aligned}
p_n(k) &= p_n(k-1) + \frac{1}{k}\Big(\ell n(1 - p_n(k-1))(p_n(k-1) - 1) \\
&\qquad + (1 - p_n(k-1)) - \ell n 2\Big) + \mathcal{O}(1/k^2).
\end{aligned} \tag{12}$$

Set $q_n(k) = p_n(k) - p^*$. Then

$$q_n(k) = \left(1 + \frac{D_1 + O(q_n(k-1))}{k}\right) q_n(k-1) + \mathcal{O}(1/k^2),$$

where $D_1 = \ell n(1-p^*)$. Since $p_n(0) \sim 3/4$ we have $q_n(0) \sim 3/4 - p^* = 0.07\cdots$. This means that it is expected that $D_1 + \mathcal{O}(q_n(k-1))$ is strictly negative for all k and sufficiently large n. Let us assume that $D_1 + \mathcal{O}(q_n(k-1)) \le -0.1$ then

$$|q_n(k)| \le \left(1 - \frac{1}{10k}\right) |q_n(k-1)| + \mathcal{O}(1/k^2) = \alpha_k |q_n(k-1)| + \beta_{n,k} \tag{13}$$

where $\alpha_k = 1 - 1/(10k)$ and $\beta_{n,k} = \mathcal{O}(1/k^2)$. Hence

$$\begin{aligned} |q_n(k)| &\le \alpha_1 \cdots \alpha_k |q_n(0)| + \alpha_2 \cdots \alpha_k \beta_{n,1} \\ &\quad + \alpha_3 \cdots \alpha_k \beta_{n,2} + \cdots + \alpha_k \beta_{n,k-1} + \beta_{n,k} \\ &= \mathcal{O}\left(\frac{1}{k^{0.1}}\left(1 + 2^{-1.9} + \cdots + k^{-1.9}\right)\right) \\ &= \mathcal{O}(k^{-1/10}), \end{aligned}$$

where we have used that

$$\alpha_i \cdots \alpha_k \le e^{-0.1(1/i + \cdots + 1/k)} = \mathcal{O}\left(\frac{i}{k}\right)^{0.1}.$$

It is now easy to check at each step that

$$D_1 + \mathcal{O}(q_n(k-1)) \le -0.1$$

is satisfied, and so we always can proceed to the next step. This leads to the first estimate $q_n(k) = p_n(k) - p^* = \mathcal{O}(k^{-1/10})$ uniformly for $k \le \sqrt{n}$.

Hence, for sufficiently large $k \ge k_0$ we have $D_1 + \mathcal{O}(q_n(k-1)) \le -1.1$. Therefore, for $k \ge k_0$

$$|q_n(k)| \le \left(1 - \frac{11}{10k}\right) |q_n(k-1)| + \mathcal{O}(1/k^2) = \alpha_k |q_n(k-1)| + \beta_{n,k}$$

where $\alpha_k = 1 - 11/(10k)$ and $\beta_{n,k} = \mathcal{O}(1/k^2)$ which leads to

$$\begin{aligned} |q_n(k)| &\le \alpha_{k_0+1} \cdots \alpha_k |q_n(0)| + \alpha_{k_0+2} \cdots \alpha_k \beta_{n,k_0+1} \\ &\quad + \alpha_{k_0+3} \cdots \alpha_k \beta_{n,k_0+2} + \cdots + \alpha_k \beta_{n,k-1} + \beta_{n,k} \\ &= \mathcal{O}\left(\frac{1}{k^{1.1}}\left(k_0^{-0.9} + \cdots + k^{-0.9}\right)\right) \\ &= \mathcal{O}(k^{-1}). \end{aligned}$$

Next set

$$u_n(k) = p_n(k) - p^* + \frac{D_2^*}{(1+D_1)k}.$$

Now, (1) resp. (12) can be rewritten to

$$u_n(k) = \left(1 + \frac{D_1}{k}\right) u_n(k-1) + \mathcal{O}(1/k^3) = \alpha_k u_n(k) + \beta_{n,k}$$

which leads to

$$\begin{aligned} u_n(k) &= \alpha_1 \cdots \alpha_k u_n(0) + \alpha_2 \cdots \alpha_k \beta_{n,1} \\ &\quad + \alpha_3 \cdots \alpha_k \beta_{n,2} + \cdots + \alpha_k \beta_{n,k-1} + \beta_{n,k} \\ &= \mathcal{O}\left(k^{D_1}\left(1 + 2^{-2-D_1} + \cdots k^{-2-D_1}\right)\right) \\ &= \mathcal{O}(k^{D_1}). \end{aligned}$$

Thus,

$$p_n(k) = p^* - \frac{D_2^*}{(1+D_1)k} + \mathcal{O}(k^{D_1}) \tag{14}$$

uniformly for $k \le \sqrt{n}$. ■

We will now indicate that the knowledge of the inital values $p_n(0)$ is not really important. In fact, $p_n([\sqrt{n}]) \sim p^*$ independently of $p_n(0) \in [0,1)$. For simplicity we assume that n is *very large* so that (1) rewrites to

$$p_n(k) \sim p_n(k-1) - \gamma_2 p_n(k-1) + 2\delta(k+1, \gamma_1) - \delta(k+1, \gamma_2) + (\gamma_2 - \gamma_1),$$

where $\gamma_1(k)$ and $\gamma_2(k)$ are defined by

$$(1-\gamma_1)^{k+1} = \frac{1}{2}, (1-\gamma_2)^{k+1} = 1 - p_n(k-1). \tag{15}$$

and $\delta(\ell, \gamma) := -\frac{1}{\ell+1}[(1-\gamma)^{\ell+1} - 1]$. Hence, we completely skip the dependence on n and consider the recurrence

$$p(k) = p(k-1) - \gamma_2 p(k-1) + 2\delta(k+1, \gamma_1) - \delta(k+1, \gamma_2) + (\gamma_2 - \gamma_1). \tag{16}$$

In fact, with help of Lemmata 4.2 and 4.3, we see that $p(k) = \lim_{n\to\infty} p_n(k)$ exactly satisfies (16).

Proposition 5.2 *Suppose that $p(k)$ satisfies (16) with $p(0) \in [0,1]$. Then $\lim\limits_{k\to\infty} p(k) = p^*$.*

Proof The unique fixed point of (16) is easily seen to be p^*. Let $\Delta_k := p(k) - p(k-1)$ and

$$f(p,k) := -\gamma_2 p + 2\delta(k+1, \gamma_1) - \delta(k+1, \gamma_2) + (\gamma_2 - \gamma_1) \tag{17}$$

It is easy to check that, as $k \to \infty$, we have

$$f \to 0, k\frac{\partial f}{\partial k} \sim \frac{-g}{k}, k\frac{\partial f}{\partial p} \to \ell n(1-p), kf \to g, g \sim \frac{\partial g}{\partial p}(p - p^*),$$

where $g := -\ell n(2) + (1-p) + (p-1)\ell n(1-p)$ and $\ell n(1-p^*) = -1 \cdot 11607666503374 \cdots = D_1$.

Moreover, $f(p,k) = g(p)/k + D_2/k^2 + \mathcal{O}(1/k^3)$. ($D_2$ is computed below). This leads to (we set $1/k = \epsilon$):

$$\begin{aligned} kf(p,k) &\sim 0 + D_1(p-p^*) \\ &+ \epsilon \lim_{\substack{\epsilon\to 0 \\ p\to p^*}} \left[[-\frac{1}{\epsilon^2}[g(p)\epsilon + D_2\epsilon^2] + \frac{1}{\epsilon}[g(p) + 2\epsilon D_2] \right] \\ &= D_1(p-p^*) + D_2^*\epsilon, \qquad \text{where} \qquad D_2^* = D_2(p^*) \end{aligned} \tag{18}$$

D_2^* is easily computed:

$$D_2 = \lim_{k\to\infty} k[kf(p,k) - g(p)] = p + \ell n(1-p)(1-p) + \ell n(2) - 1 + \ell n^2(2)/2 + \ell n^2(1-p)\frac{1}{2}(p-1)$$

and

$$D_2^* = \ell n^2(2)/2 + \ell n^2(1-p^*)\frac{1}{2}(p^*-1) = \cdot 036216193421872 \cdots$$

(16) leads to

$$k\Delta_k \sim D_1(p-p^*) + D_2^*/k \tag{19}$$

in the neighbourhood of p^*.

As k is large, the solution of (19) can be approximated by the solution of the corresponding ODE (ordinary differential equation). We obtain $kp'_k = D_1(p-p^*) + D_2^*/k$ which is satisfied by

$$p \sim p^* + \frac{k^{D_1}p_1}{k_1^{D_1}} - \frac{D_2^*}{k(1+D_1)} + \frac{k^{D_1}D_2^*}{k_1(1+D_1)k_1^{D_1}} \tag{20}$$

if the initial condition is $p(k_1) = p_1 + p^*$.

Fig. 1 displays the fact that the convergence is independent of $p \in [0,1)$; we fall rapidly within the domain of attraction of p^*. Fig. 1 gives $p(k)$ for $p(0) = \{0.0, 0.25, 0.50, 0.75, 1.00\}$. This set of starting values will be called "typical set" in the sequel.

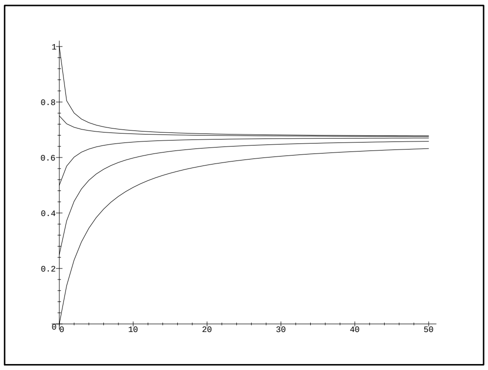

Figure 1 : $p(k)$

Asymptotically, we have $\Delta_k \sim D_3/k^2$ for some D_3.

However the convergence of $p(k)$ to p^* is not necessarily monotonous: Fig. 2 shows that, for $p(0) = 0.5$, $p(k) - p(k-1)$ changes its sign in the neighbourhood of $[p = \cdot 67245, k = 1974]$. This is due to the fact that $f(p,k)$ (see (17)) becomes 0 there. The implicit curve $\mathcal{C} : f(p,k) = 0$ is plotted in Fig. 2.

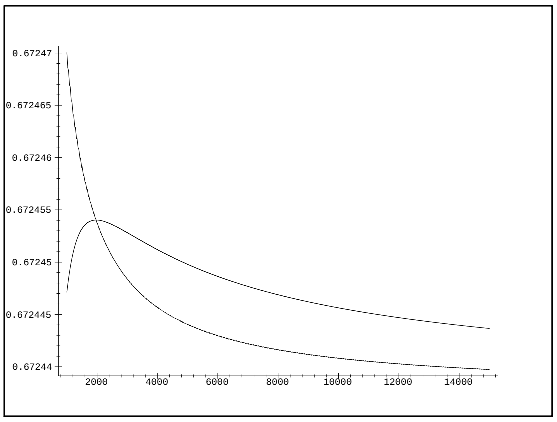

Figure 2: p and $\mathcal{C} = f(p,k) = 0, p(0) = \cdot 50$.

So we must carefully analyze the possible intersections of $p(k)$ and $\mathcal{C}$. $f(p,1)$ gives the initial slope of $p(k)$. $f(p,1) = 0$ for $p = p_3 = \cdot 68925652839617 \cdots$ which is the upper bound for a starting point leading to a behaviour like Fig. 2.

The lower bound p_2 is actually 0. Indeed (18) show that $\mathcal{C}$ is asymptotically given by

$$p(k) \sim p^* - \frac{D_2^*}{D_1 k}.$$

The intersection with $p(k)$ as given by (20) leads to an equation, the unique solution $\tilde{k}$ of which is (after plugging all constants in), for $k_1 = 15000$:

$$\begin{aligned} \ell n(\tilde{k}) \ &\sim \ 10 \cdot 9803644415464 \cdots \\ &+ \ 8 \cdot 61499595727815 \cdots \ell n(-45797.102862568 \cdots p_1 \\ &+ \ \cdot 952586944700164 \cdots) \end{aligned} \tag{21}$$

where again $p_1 := p(k_1) - p^*$.

It appears that, for $p(0) \in [0, p_3]$, we have $p_1 \in [-6\ 10^{-5}, 2\ 10^{-5}]$ (this is computed numerically with (16), see Fig. 3). We have done a numerical check of (21), which is indeed quite precise. This shows that only one intersection exists between $p(k)$ and $\mathcal{C}$ for $p(0) \in [0, p_3]$. After the intersection, the convergence is monotonous. Outside this range, i.e. for $p(0) \in [p_3, 1]$, no intersection do exist and the convergence is always monotonous. In both cases, we fall within the domain of attraction of p^* and our convergence analysis can be applied (as given by (20)). ■

Fig. 3 gives p for the typical set and for large values of k.

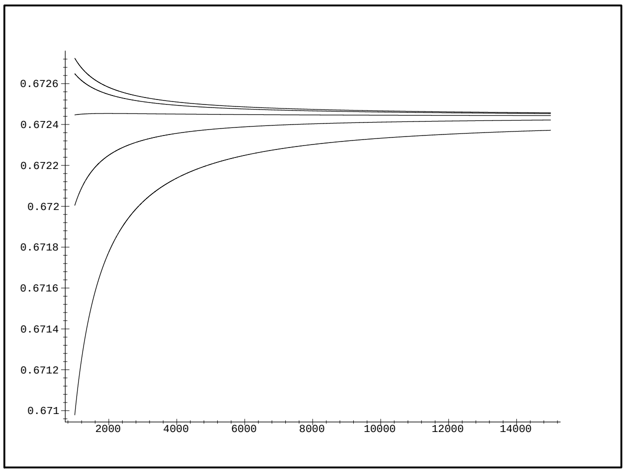

Figure 3: $p(k)$ for large k

5.2 Subrange 2: $n^{1/2} \le k \le n^{3/4}$

Proposition 5.3 *We have*

$$p_n(k) = p^* + p'\frac{k}{n} + \mathcal{O}\left(\frac{1}{k}\right) + \mathcal{O}\left(\left(\frac{k}{n}\right)^2\right). \tag{22}$$

uniformly for $\sqrt{n} \le k \le \epsilon n$, *where* $\epsilon > 0$ *is a sufficiently small constant.*

Proof We show that is possible to use (14) for $k_0 = [\sqrt{n}]$ as initial values in order to obtain a first estimate for $p_n(k)$ if $k \le \epsilon n$ (for some given small $\epsilon > 0$). By using similar approximations as above (e.g. we have to use the second approximation term for $C(n, k, r)$ in Lemma 4.2) we obtain for $q_n(k) = p_n(k) - p^*$ and $k \ge \sqrt{n}$

$$q_n(k) = \left(1 + \frac{D_1 + O(q_n(k-1))}{k}\right) q_n(k-1) + \mathcal{O}(1/n). \tag{23}$$

The only difference to the above approximation (13) is that the error term $\mathcal{O}(1/k^2)$ is replaced by the worse error term $\mathcal{O}(1/n)$ (recall that $k \ge \sqrt{n}$). We again assume that $D_1 + \mathcal{O}(q_n(k-1)) \le -0.1$, and it will turn out that this assumption is satisfied is $k \le \epsilon n$, where $\epsilon > 0$ is chosen sufficiently small. Set $k_0 = [\sqrt{n}]$. For $k \ge k_0$ we have

$$|q_n(k)| \le \left(1 - \frac{1}{10k}\right) |q_n(k-1)| + \mathcal{O}(1/k^2) = \alpha_k |q_n(k-1)| + \beta_{n,k}$$

where $\alpha_k = 1 - 1/(10k)$ and $\beta_{n,k} = \mathcal{O}(1/n)$. Hence,

$$\begin{aligned}
|q_n(k)| &\leq \alpha_{k_0+1}\cdots\alpha_k|q_n(k_0)| + \alpha_{k_0+2}\cdots\alpha_k\beta_{n,k_0+1} \\
&\quad +\alpha_{k_0+3}\cdots\alpha_k\beta_{n,k_0+2} + \cdots + \alpha_k\beta_{n,k-1} + \beta_{n,k} \\
&= \mathcal{O}(1/k_0) + \mathcal{O}\left(\frac{1}{nk^{0.1}}\left(k_0^{0.1} + (k_0+1)^{0.1} + \cdots + k^{0.1}\right)\right) \\
&= \mathcal{O}(1/k_0) + \mathcal{O}(k/n) = \mathcal{O}(k/n).
\end{aligned}$$

Again it is possible to use this estimate $q_n(k) = p_n(k) - p^* = \mathcal{O}(k/n)$ as the starting point for a bootstrapping procedure. By using Lemma 4.3 and similar estimates as in the proof of Lemma 4.4 we obtain

$$q_n(k) = \left(1 + \frac{F(q_n(k-1))}{k}\right) q_n(k-1) + \frac{G(q_n(k-1))}{n} + \mathcal{O}\left(\frac{1}{k^2}\right) + \mathcal{O}\left(\frac{k}{n^2}\right)$$

and for $v_n(k) := p_n(k) - p^* - p'\frac{k}{n}$

$$v_n(k) = \left(1 + \frac{F(r_n(k-1))}{k}\right) v_n(k-1) + \frac{G_1(r_n(k-1))v_n(k-1)}{n} + \mathcal{O}\left(\frac{1}{k^2}\right) + \mathcal{O}\left(\frac{k}{n^2}\right).$$

Since we know a-priori that $v_n(k) = \mathcal{O}\left(\frac{k}{n}\right)$ we also have for $\sqrt{n} \leq k \leq \epsilon n$ (where $\epsilon > 0$ is sufficiently small)

$$|v_n(k)| \leq \left(1 - \frac{11}{10k}\right)|v_n(k-1)| + \mathcal{O}\left(\frac{1}{k^2}\right) + \mathcal{O}\left(\frac{k}{n^2}\right) = \alpha_k|v_n(k-1)| + \beta_{n,k}$$

which leads to

$$\begin{aligned}
|v_n(k)| &\leq \alpha_{k_0+1}\cdots\alpha_k|v_n(k_0)| + \alpha_{k_0+2}\cdots\alpha_k\beta_{n,k_0+1} \\
&\quad +\alpha_{k_0+3}\cdots\alpha_k\beta_{n,k_0+2} + \cdots + \alpha_k\beta_{n,k-1} + \beta_{n,k} \\
&= \mathcal{O}\left(\left(\frac{k_0}{k}\right)^{1.1}\frac{1}{k_0}\right) + \mathcal{O}\left(\frac{1}{k^{1.1}}\left(k_0^{-0.9} + (k_0+1)^{-0.9} + \cdots + k^{-0.9}\right)\right) \\
&\quad +\mathcal{O}\left(\frac{1}{n^2k^{1.1}}\left(k_0^{2.1} + (k_0+1)^{2.1} + \cdots + k^{2.1}\right)\right) \\
&= \mathcal{O}\left(\frac{1}{k}\right) + \mathcal{O}\left(\left(\frac{k}{n}\right)^2\right)
\end{aligned}$$

This proves (22). ■

Setting $p(\epsilon) := \lim_{n\to\infty} p_n(\epsilon n)$ we obtain again with Lemmata 4.2 and 4.3 a limiting ODE

$$\frac{dp(\epsilon)}{d\epsilon} = \frac{1}{1-\epsilon}[-r_2p(\epsilon) + \frac{1-\epsilon}{\epsilon} - \frac{p(\epsilon)(1-\epsilon)}{\epsilon} + (r_2 - r_1)]. \tag{24}$$

with $(1-\epsilon)^{r_1} = 1/2, (1-\epsilon)^{r_2} = 1 - p(\epsilon)$. The RHS of this equation has a singularity at $\epsilon = 0$, but it is easily seen that this singularity is removed if we take $p(0) = p^*$ (which is of course expected). We could not solve (24) explicitly but a solution in series of ϵ is easily computed. We have used in the sequel a series up to 15 terms, starting with (from now on, we use a precision of 15 digits):

$$p(\epsilon) = p^* + p'\epsilon - \epsilon^2 \cdot 0452929692511657 \cdots + \cdots \tag{25}$$

and the coefficient of $\epsilon^{15} \sim -6.10^{-3}$.

A simulation of $p_n(k)(n = 300, \text{with } p_n(0) = 0.75, k \in [0 \cdot\cdot 190])$ is given in Fig. 4 together with the solution of (24).

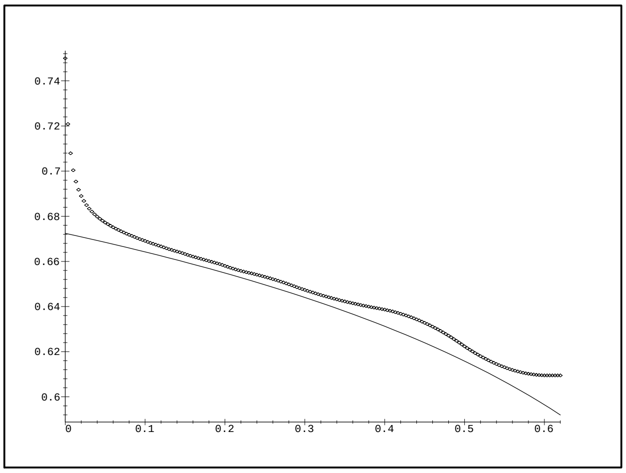

Figure 4: $p_n(k)$ and $p(\epsilon)$

The similarity is already surprisingly good ! Of course $p_n(k)$ becomes stationary for some value of k, k^* say. The stationary value $p_n(k^*)$ is given by $0.6094\cdots$

5.3 Subrange 3: $n^{3/4} \leq k \leq 3n/4$

This subrange is the most interesting one. The idea is to compare $p_n(k)$ with $p(k/n)$, where $p(s)$ is the *solution* $p(s)$, $s > 0$, of the differential equation (2). We will prove the following property:

Proposition 5.4 *We have*

$$p_n(k) - p(k/n) = \mathcal{O}(1/\sqrt{n}) \tag{26}$$

uniformly for $n^{1/2} \leq k \leq 3n/4$ *as* $n \to \infty$.

Proof The idea is to prove this step by step. By Proposition 5.3 and (10) it follows that (26) is surely true for $n^{1/2} \leq k \leq n^{3/4}$.

Let $k_i(n)$ be defined by

$$C(n, k_i(n), i) \geq \frac{1}{2}, \quad C(n, k_i(n), i+1) < \frac{1}{2}$$

and $\tilde{k}_j(n)$ by

$$C(n, \tilde{k}_j(n), j) \geq 1 - p_n(k_i(n) - 1), \quad C(n, \tilde{k}_j(n), j+1) < 1 - p_n(k_i(n) - 1).$$

Then it is easy to show that

$$k_i(n) = n\beta_i + \mathcal{O}(1).$$

Furthermore, if we assume that (26) holds then

$$\tilde{k}_j(n) = n\tilde{\beta}_j + \mathcal{O}(n^{1/2}/j).$$

(Note that $\tilde{\beta}_j$ is defined by $(1 - \tilde{\beta}_j)^j = 1 - p(\tilde{\beta}_j)$.)

Our aim is to show inductively

$$p_n(\tilde{k}_j(n)) = p\left(\frac{\tilde{k}_j(n)}{n}\right) + \mathcal{O}(n^{-1/2}), \tag{27}$$

where $j_0 \sim n^{3/4}$. (As mentioned above (27) is true for $j = j_0$ by Proposition 5.3 and (10).)

In order to demonstrate the method let us suppose that we already know that (27) holds for some $j < j_0$ and that

$$k_{i+1}(n) \leq \tilde{k}_{j+1}(n) \leq \tilde{\beta}_j n < \tilde{k}_j(n) \leq k_i(n),$$

i.e. there is no $k_i(n)$ contained in the interval $(\tilde{k}_{j+1}(n), \tilde{k}_j(n))$. We now consider the original recurrence for $p_n(k)$ for $\tilde{k}_{j+1}(n) < k \leq \tilde{k}_j(n)$. We have

$$p_n(k) = \frac{1}{n} f_{i,j}\left(\frac{k+1}{n}\right)(1 + \mathcal{O}(j/n)) + \left(1 - \frac{j}{n - (k+1)}\right) p_n(k-1). \tag{28}$$

Setting

$$q_n(k) = p_n(k) \exp\left(\sum_{l=k_{j+1}(n)}^{k} \frac{j}{n - (l+1)}\right) = p_n(k)\left(\frac{1 - \frac{k_{j+1}(n)}{n}}{1 - \frac{k}{n}}\right)^j (1 + \mathcal{O}(j/n))$$

we immediately can rewrite (28) as

$$q_n(k) - q_n(k-1) = \frac{1}{n}\left(\frac{1 - \frac{k_{j+1}(n)}{n}}{1 - \frac{k}{n}}\right)^j f_{ij}\left(\frac{k+1}{n}\right)(1 + O(j/n)).$$

Hence,

$$\begin{aligned}
q_n(k_j(n)) &= q_n(k_{j+1}(n)) + \sum_{k=k_{j+1}(n)+1}^{k_j(n)} \frac{1}{n}\left(\frac{1 - \frac{k_{j+1}(n)}{n}}{1 - \frac{k}{n}}\right)^j f_{i,j}\left(\frac{k+1}{n}\right)(1 + \mathcal{O}(j/n)) \\
&= q_n(k_{j+1}(n)) + \int_{k_{j+1}(n)/n}^{k_j(n)/n} \left(\frac{1 - k_{j+1}(n)/n}{1 - t}\right) f_{i,j}(t)\, dt + \mathcal{O}(j/n).
\end{aligned}$$

Here we have used that the integrand is monotonic and that in this case the difference between sum and integral can be estimated by the maximum of the integrand (which is $\mathcal{O}(j)$). Thus, we can compare $q_n(k_j(n)) - q_n(k_{j+1}(n))$ and $q(k_j(n)/n) - q(k_{j+1}/n)$. We only have to take into account that (usually) $\tilde{k}_{j+1}(n) \neq \tilde{\beta}_j n$. We proceed similarly to the proof of Lemma 4.7. If we assume that $|p_n(k) - p(k/n)| \leq D/\sqrt{n}$ then

$$|\tilde{k}_{j+1}(n) - \tilde{\beta}_j n| \leq \frac{(D+\varepsilon)\sqrt{n}}{(1-p^*)j}$$

and we can proceed as above. Note that we can choose D large enough such that we always can justify the a-priori estimate $|p_n(k) - p(k/n)| \leq D/\sqrt{n}$. This completes the proof of the Proposition 5.4. ■

6 The Algorithm

In this section we present a computational algorithms in order to calculate $p(\frac{3}{4})$ of the solution of (2).

6.1 Differential equations

It is clear that we have to treat an (infinite) sequence of ODE, depending on the ranges defined by $(1-s)^i = \frac{1}{2}, (1-s)^j = 1 - p(s)$, i and j integers. As in Sec. 4.2, we obtain

$$\frac{dp(s)}{ds} = \frac{1}{1-s}[-r_2 p(s) + 2\sigma(r_1, s) - \sigma(r_2, s) + (r_2 - r_1)] \tag{29}$$

where $\sigma(r,s) := -[(1-s)^r - 1](1-s)/s$.

Set $\beta_i := 1 - \frac{1}{2^{1/i}}$ and $\tilde{\beta}_i = 1 - [1 - p(\tilde{\beta}_i)]^{1/i}$. The following table gives the first values of β_i (3 digits) and an estimation of $\tilde{\beta}_i$ based on our simulation ($n = 300$):

i	β_i	$\tilde{\beta}_i$ (estimated)
1	·500	·61
2	·293	·40
3	·206	·30
4	·159	·24
5	·129	·20

(29) gives a different ODE in each subrange delimited by two values of the set $\{\beta_i, \tilde{\beta}_j\}$. For instance the last equation, for $s \geq \frac{1}{2}$, corresponds to $r_1 = 0, r_2 = 1$ and is given by

$$\frac{dp}{ds} = \frac{-p+s}{1-s}, \tag{30}$$

the solution of which is

$$p(s) = -s\ \ell n(1-s) - \ell n(2)s + 2s - 2s\ a + \ell n(1-s) + \ell n(2) - 1 + 2a, \quad \text{if } p(\frac{1}{2}) = a.$$

Of course, the stationary solution of (30) occurs when $p(s^*) = s^*$ (i.e. $\tilde{\beta}_1$) and s^* is the solution of our initial problem. r_2 becomes 0 at s^*. The penultimate equation is given for $s \in [\tilde{\beta}_2, \frac{1}{2}] (r_1 = r_2 = 1)$ by

$$\frac{dp}{ds} = \frac{-p(s) + 1 - s}{1 - s}.$$

Actually each ODE in (29) has an explicit solution, immediately found by MAPLE. Looking at numerical values of β_i and $\tilde{\beta}_j$, it is easily seen that the β_i values are always separated by some $\tilde{\beta}_j$ values and of course the set $\{\tilde{\beta}_j\}$ is more dense than the set $\{\beta_i\}$. When $s \in [\beta_i, \beta_{i-1}], r_1 = i-1$ and when $s \in [\tilde{\beta}_j, \tilde{\beta}_{j-1}], r_2 = j-1$. $p(s)$ is of course continuous by construction, but $p'(s)$ is also continuous. Indeed, at β_i^-, the RHS of (29) gives

$$[-i + 2\frac{(1-s)}{2s} + f_1(s)]_{\beta_i} \qquad \text{for some } f_1, \tag{31}$$

and at β_i^+, we obtain

$$[-(i-1) - 2\frac{(2s-1)}{s2(1-s)}(1-s) + f_1(s)]_{\beta_i}$$

which is identical to (31). Similarly, at $\tilde{\beta}_j^-$, we derive

$$[-p(s)j + j + \frac{-p(s)(1-s)}{s} + f_2(s)]_{\tilde{\beta}_j} \qquad \text{for some } f_2, \tag{32}$$

and at $\tilde{\beta}_j^+$, we obtain

$$[-p(s)(j-1) + j - 1 + \frac{(s-p(s))(1-s)}{s(1-s)} + f_2(s)]_{\tilde{\beta}_j}$$

which is identical to (32).

Let us now analyze $p''(s)$. From (29), we derive

$$\begin{aligned} p''(s) &= \frac{p'(s)}{1-s} + \Big\{-p'(s)r_2 + \frac{2}{s^2}[-s[-(r_1+1)(1-s)^{r_1} + 1] + [(1-s)^{r_1} - 1](1-s)] \\ &- \frac{1}{s^2}[-s[-(r_2+1)(1-s)^{r_2} + 1] + [(1-s)^{r_2} - 1](1-s)]\Big\} / (1-s). \end{aligned}$$

At β_i, we compute (omitting the details)

$$\Delta_i := p''(\beta_i^+) - p''(\beta_i^-) = \frac{i}{(1-\beta_i)^2}.$$

At $\tilde{\beta}_j$, we obtain

$$\Delta_j := p''(\tilde{\beta}_j^+) - p''(\tilde{\beta}_j^-) = \frac{1}{1-\tilde{\beta}_j}[p'(\tilde{\beta}_j) - j\frac{[1-p(\tilde{\beta}_j)]}{1-\tilde{\beta}_j}].$$

We see that $p''(s)$ is discontinuous at β_i and $\tilde{\beta}_j$, $p''(s)$ is only piecewise continuous.

6.2 Computation algorithm

The numerical strategy is now the following: we proceed from 0 to some β_i with $p(\epsilon)$ as given in subrange 2. We set $r_1 := i-1$. From β_i, with initial value $p(\beta_i)$, we start with an ODE given by (29). We compute j such that $\tilde{\beta}_{j+1} < \beta_i \leq \tilde{\beta}_j$. We set $r_2 = j$. ¿From now on, we compute (with MAPLE) each solution of (29) and proceed to the shortest next value ($\beta.$ or $\tilde{\beta}.$). $\tilde{\beta}$ is of course computed with the new solution $p(s)$. If the next value is $\beta.$, set $r_1 := r_1 - 1$ otherwise set $r_2 := r_2 - 1$. Stop as soon as $r_2 = 0$ (see the discussion above). ■

We have experimented with starting values i given in the following table, together with the final solution s^* (we give only the 8 first significant digits):

i	s^*
7	$\cdot$ 60857830
10	$\cdot$ 60859431
12	$\cdot$ 60859791
15	$\cdot$ 60860021
18	$\cdot$ 60860123
24	$\cdot$ 60860203
27	$\cdot$ 60860220
30	$\cdot$ 60860231

We noted that 15 digits of precision will not be sufficient after $i = 30$. Interpolating the set $[\beta(i), s^*(i)]$ with a cubic curve gives the following Fig. 5, leading to $s^* = \cdot 6086026\cdots$

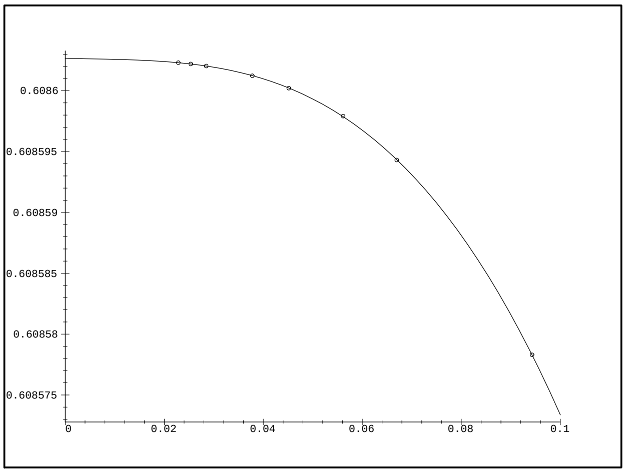

Figure 5: Interpolation

If we constraint the cubic to avoid any linear term, (according to Lemma 4.7), we obtain actually nearly the same curve, with $s^* = \cdot 6086025\cdots$

Our last solution ($i = 30$) corresponds to 77 successive ranges for s. The comparison with the simulation ($n = 300$) is given in Fig. 6.

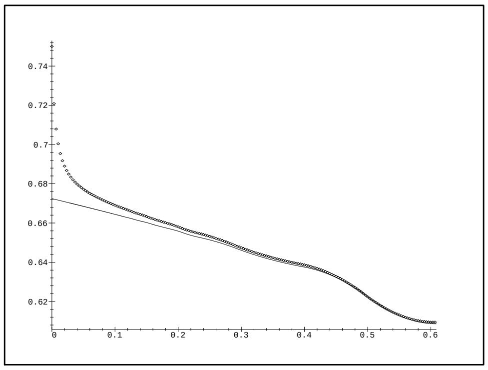

Figure 6: Computed and Simulated solution

We have also computed $p'(s)$ and simulated values given by $p_n(k) - p_n(k-1)$. These are given in Fig. 7, when we see clearly the convergence of our numerical procedure near the origin (the short straight line is computed according to the subrange 2 technique).

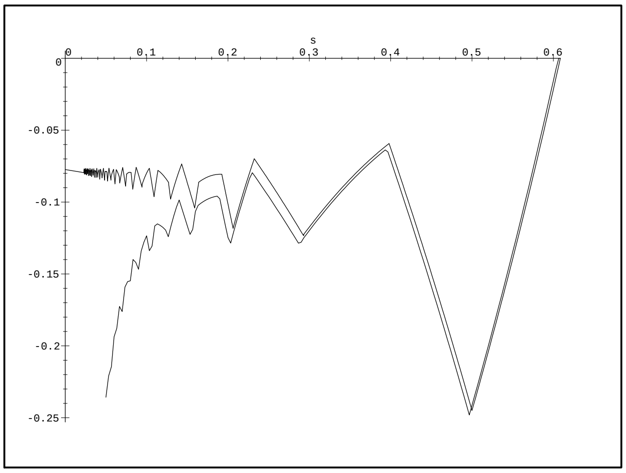

Figure 7: Computed and Simulated $p'(s)$

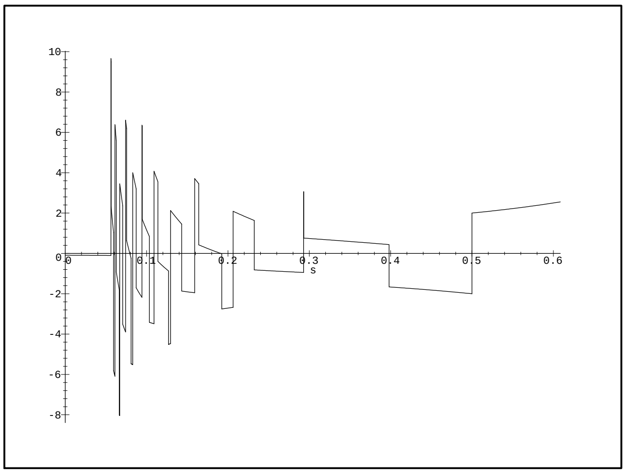

Figure 8: Computed $p''(s)$

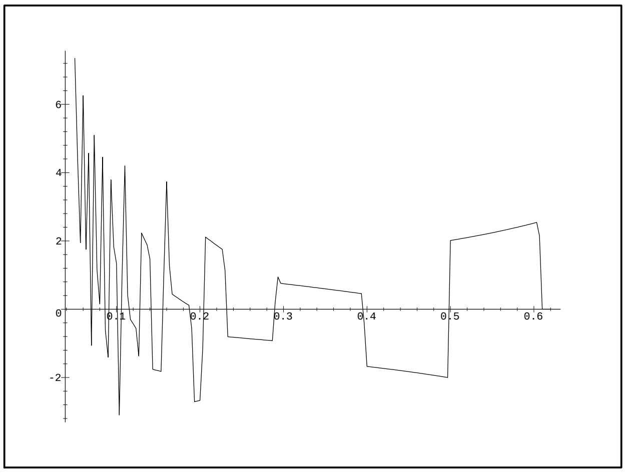

Figure 9: Simulated $p''(s)$

Computed (starting value $i = 12$) and simulated (with second order divided differences) values for $p''(s)$ are given in Fig. 8 and 9.

6.3 Some asymptotic properties

We will now obtain some properties of $\beta_i, \tilde{\beta}_j$ and $p(\beta_i), p(\tilde{\beta}_j)$. The proofs are rather technical and will be omitted. They use (10).

First of all, we must compute r_1 (related to β_i) and r_2 (related to $\tilde{\beta}_j$ say). The solution of (29) can be interpreted as an analytic function for $s \in [\beta_i, \tilde{\beta}_j]$, so we set

$$p(s) = p(\beta_i) + q' \cdot (s - \beta_i) + \frac{q''}{2}(s - \beta_i)^2 + \Theta(s - \beta_i)^3.$$

To compute r_2, which must be integer, we first compute some real $\bar{r}_2$ such that $(1-\beta_i)^{\bar{r}_2} = 1 - p(\beta_i)$. This leads to

$$\bar{r}_2(\beta_i) = \frac{-\ell n(1-p^*)}{\beta_i} + \bar{r}_{20} + \mathcal{O}(\beta_i) \tag{33}$$

with (we set $p' = p'(0)$ as given by (6))

$$\bar{r}_{20} = [p' + \frac{1}{2}\ell n(1-p^*)(1-p^*)]/(1-p^*) = -\cdot 794324667630359\cdots$$

r_2 must be such that $r_2 - 1 \leq \bar{r}_2 < r_2$ so $r_2 = \bar{r}_2 - C_1(\beta_i)$ for some $0 < C_1(\beta_i) \leq 1$ *different in each subrange.* r_1 is given by $-\ell n(2)/\ell n(1-\beta_i) - 1$. Note that $r_1 = \frac{\ell n(2)}{\beta_i} + \Theta(1)$.

We now use (29) to compute q' and q''. q' is given by (we use (4) here): $q' = p' + \mathcal{O}(\beta_i)$.
$q'' = D_4/\beta_i + \mathcal{O}(1)$ for some D_4 linear in $C_1(\beta_i)$. D_4 is a rather complicated expression, the numerical value of which is

$$D_4 = -\cdot 264682986785600\cdots - \cdot 182792383291221\cdots C_1(\beta_i)$$

This confirms the high irregularity of $p''(s)$.

We shall see in the sequel that, asymptotically, as $i \to \infty$, between two values $[\beta_i, \beta_{i-1}]$ one or two values of $\tilde{\beta}_j$ occur. (The analysis is similar for one intermediate $\tilde{\beta}_j$) .
Let us now analyze the sizes of each subrange $[\beta_i, \tilde{\beta}_j], [\tilde{\beta}_j, \tilde{\beta}_{j-1}], [\tilde{\beta}_{j-1}, \beta_{i-1}]$, which are determined by the following lemma:

Lemma 6.1 *Suppose that $\beta_i < \tilde{\beta}_j < \tilde{\beta}_{j-1} < \beta_i$. Then*

$$\tilde{\beta}_j - \beta_i = -C_1(\beta_i)\beta_i^2/\ell n(1-p^*) + \mathcal{O}(\beta_i^3)$$

where $C_1(\beta_i)$ is defined above.
Similarly, we have

$$\tilde{\Delta}_j := \tilde{\beta}_{j-1} - \tilde{\beta}_j = -\frac{\tilde{\beta}_j^2}{\ell n(1-p^*)} + \frac{\tilde{\beta}_j^3}{\ell n^2(1-p^*)}\beta_i^3 + \mathcal{O}(\tilde{\beta}_j^4),$$

another form is given by

$$\begin{aligned}
\tilde{\Delta}_j &= -\frac{\beta_i^2}{\ell n(1-p^*)} + \frac{2C_1(\beta_i)+1}{\ell n^2(1-p^*)}\beta_i^3 + \mathcal{O}(\beta_i^4), \\
\beta_{i-1} - \tilde{\beta}_{j-1} &= -\frac{C_2(\beta_i)\beta_{i-i}^2}{\ell n(1-p^*)} + \mathcal{O}(\beta_{i-1}^3)
\end{aligned}$$

Furthermore we have

$$C_2(\beta_i) = \begin{cases} \rho_0(\beta_i) - 1 - C_1(\beta_i) & \textit{(two intermediate } \tilde{\beta}_j\textit{) or} \\ \rho_0(\beta_i) - C_1(\beta_i) & \textit{(one intermediate } \tilde{\beta}_j\textit{),} \end{cases}$$

where

$$\rho_0(\beta_i) := \bar{r}_2(\beta_{i-1}) - \bar{r}_2(\beta_i).$$

Finally

$$\Delta_i := \beta_{i-1} - \beta_i = \frac{\beta_i^2}{\ell n(2)} + \beta_i^3/\ell n^2(2) + \mathcal{O}(\beta_i^4).$$

Remark : Since $\rho := \frac{-\ell n(1-p^*)}{\ell n(2)} = 1 \cdot 6101582698607 \cdots$ we immediately check that, asymptotically as $i \to \infty$, between two values β_i and β_{i-1}, one or two values of $\tilde{\beta}_j$ occur. In the first case, $C_2(\beta_i) = \rho_0(\beta_i) - C_1(\beta_i)$, in the other case, $C_2(\beta_i) = \rho_0(\beta_i) - 1 - C_1(\beta_i)$.

If we set $q := 1 - p^*$, we see that q satisfies the equation:

$$e^q = 2q^q \tag{34}$$

and $\rho = \frac{1}{q} - \frac{1}{\ell n(2)}$. ρ itself satisfies the equation

$$\rho = 2^\rho - \frac{1}{\ell n(2)}. \tag{35}$$

Hence, ρ is irrational. ■

Similarly to lemma 6.1, there is of course a relation between $C_1(\beta_i), C_2(\beta_i)$ and $\beta., \tilde{\beta}.$. This is given in the following lemma:

Lemma 6.2 *We have*

$$\begin{array}{rrcll} & C_1(\beta_i) & = & \dfrac{\tilde{\beta}_j - \beta_i}{\tilde{\Delta}_j} + \mathcal{O}(\beta_i) & \\ & C_2(\beta_i) & = & \dfrac{\beta_{i-1} - \tilde{\beta}_{j-1}}{\tilde{\Delta}_j} + \mathcal{O}(\beta_i) & \text{2 intermediate points} \\ \text{or} & C_2(\beta_i) & = & \dfrac{\beta_{i-1} - \tilde{\beta}_j}{\tilde{\Delta}_j} + \mathcal{O}(\beta_i) & \text{1 intermediate point} \end{array}$$

with $\tilde{\Delta}_j := \tilde{\beta}_{j-1} - \tilde{\beta}_j$.
We also have $\rho_1(\beta_i) := \frac{\beta_{i-1}-\beta_i}{\tilde{\Delta}_j} = \Delta_i/\tilde{\Delta}_j = \rho + \mathcal{O}(\beta_i)$ *and* $\rho_o(\beta_i) = \rho + \mathcal{O}(\beta_i)$.

We now turn to $p'(\tilde{\beta}_j) - p'(\beta_i)$ and the other corresponding differences which we need. We obtain the following result.

Proposition 6.3 *We have*

$$p'(\tilde{\beta}_j) - p'(\beta_i) = [C_9 C_1(\beta_i) + C_{10} C_1^2(\beta_i)]\beta_i + \mathcal{O}(\beta_i^2)$$

with

$$\begin{aligned} C_9 &:= \frac{1}{2}(p^* + 2\ell n(2) - 1)(1 - p^*)/D_6 \\ &= \cdot 474309686920287 \cdots \end{aligned}$$

where
$$D_6 := p^* + \ell n(2) - 1 = \cdot 365584766582428 \cdots$$
and
$$\begin{aligned} C_{10} &:= \frac{1}{2}(1-p^*)(2p^* - 2p^*\ell n(2) + p^*\ell n^2(2) - 1 - 2\ell n^2(2) + 2\ell n(2) - {p^*}^2)/D_6^2 \\ &= - \cdot 356637080685928 \cdots \end{aligned}$$
Similarly,
$$p'(\beta_{i-1}) - p'(\beta_i) = [C_5 + C_6 C_1(\beta_i)]\beta_i + \mathcal{O}(\beta_i^2),$$
with two intermediate values $\tilde{\beta}_j, \tilde{\beta}_{j-1} \in [\beta_i, \beta_{i-1}]$, *and*
$$p'(\beta_{i-1}) - p'(\beta_i) = [C_7 + C_8 C_1(\beta_i)]\beta_i + \mathcal{O}(\beta_i^2)$$
with one intermediate value $\tilde{\beta}_j \in [\beta_i, \beta_{i-1}]$,
where
$$\begin{aligned} C_5 &:= -\frac{1}{2}D_7[\ell n(2) + 1 - p^* - 2p^*\ell n(2)]/[\ell n^2(2)(1-p^*)] \\ &= - \cdot 0389578422803963 \cdots \\ C_6 &:= [\ell n(2) + 1 - p^* - 2p^*\ell n(2)]/\ell n(2) \\ &= \cdot 127697498182012 \cdots \\ C_7 &:= \frac{1}{2}D_6 D_7/[\ell n^2(2)(1-p^*)] \\ &= \cdot 160907073515105 \cdots \\ C_8 &:= -\frac{D_7}{\ell n(2)} = - \cdot 199864915795502 \cdots \\ D_7 &:= p^* + p^*\ell n(2) - 1 \end{aligned}$$

Further we have
$$\begin{aligned} p'(\beta_{i-1}) - p'(\tilde{\beta}_{j-1}) &= [-C_9 C_2(\beta_i) - C_{10} C_2^2(\beta_i)]\beta_{i-1} + \mathcal{O}(\beta_{i-1}^2), \\ p'(\tilde{\beta}_{j-1}) - p'(\tilde{\beta}_j) &= [C_3 + C_4 C_1(\beta_i)]\beta_i + \mathcal{O}(\beta_i^2), \end{aligned}$$
where
$$\begin{aligned} C_3 &:= -\frac{1}{2}\ell n(2)[\ell n(2) + \ell n(1-p^*)]/\ell n^2(1-p^*) \\ &= \cdot 117672606234358 \cdots \\ C_4 &:= -\ell n^2(2)/\ell n^2(1-p^*) \\ &= - \cdot 385711747394331 \cdots \end{aligned}$$

The following figures show a confirmation of our results. Fig. 10 gives $\ell n[|p'(x_i) - p'(x_{i+1})|]$ vs $\ell n(x_i)$ and the line $z - 1.6$, where $[x_i \cdot \cdot x_{i+1}]$ are any of the subranges described in lemma 6.1. The linearity of the bounds is quite evident and confirms Proposition 6.3. Fig. 11 shows $\ell n[|p'(x_i) - p'(0) - x_i p''(0)|]$ vs $\ell n(x_i)$ and the line $z - 2.1$ where $p'(0)$ and $p''(0)$ are given in Sec. 5.2. Again the linearity of the bounds confirms (10)

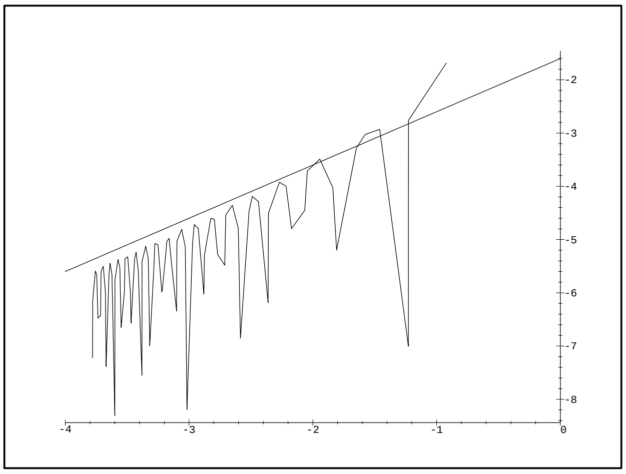

Figure 10: $\ell n[|p'(x_i) - p'(x_{i+1})|]$ vs $\ell n(x_i)$

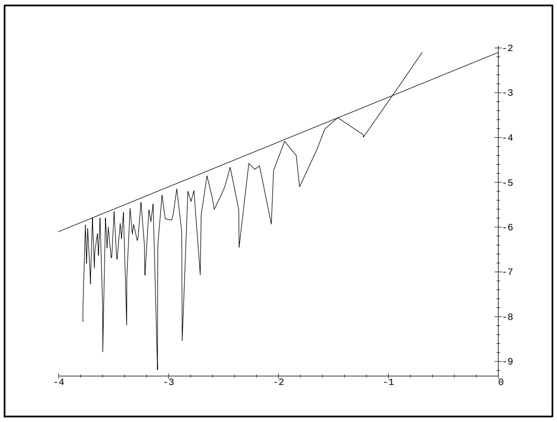

Figure 11: $\ell n[|p'(x_i) - p'(0) - x_i p''(0)|]$ vs $\ell n(x_i)$

7 Conclusion

The optimal strategies for A and B are now summarized: as soon as one of the two decision makers stops the optimal strategy of the opponent is trivial, namely to wait for a better rank, or to stop, necessarily, on option number n. If none has stopped before, A and B must apply a threshold strategy each. A must stop if and only if the relative rank r_k of the currently observed option number k satisfies $\begin{pmatrix} k \\ r_k \end{pmatrix} \geq \frac{1}{2} \begin{pmatrix} n \\ r_k \end{pmatrix}$.

For B the optimal strategy is a more complicated threshold rule, namely to accept option number k if its relative rank r_k satisfies $\begin{pmatrix} k \\ r_k \end{pmatrix} \geq (1 - p(n,k)) \begin{pmatrix} n \\ r_k \end{pmatrix}$, where the $p(n,k)$ can be recursively computed by backward induction. All this follows, in its essence, already from Enns and Ferenstein [6].

The most important gap was the question whether $p = \lim_{n\to\infty} p(n,1) = \lim_{n\to\infty} p_n(n-2)$ exists.

The answer is, as we have shown, in the affirmative, with $p = s^* = 0 \cdot 6086026 \cdots$ Moreover, we even have that for any $k = k(n) < (1-p)n$ the limit $\lim_{n\to\infty} p(n, k(n))$ is the same. Finally, $1/2$ and $3/4$ are the uniform lower and upper bound for the winning probabilities of A at different steps, and for different n, and these bounds are sharp.

The solution of the problem is now complete.

An outlook: The present paper has used a balanced mixture of probabilistic arguments, analysis, and computer algebra. As we have pointed out computer algebra played the important role to show us, on the one hand, how to assemble the proof of convergence and, on the other hand, to obtain the limiting optimal threshold strategy.

Perhaps computer algebra could play a similar role in other limit problems as e.g. in the well-known problem of minimizing the expected rank of a selected option in the full-information case. (H. Robbins worked on this problem already 35 years ago, and it is not yet completely solved; see Bruss and Ferguson ([4]) for more details and further references).

The problem is as follows: n independent identically distributed continuous random variables are observed sequentially. A decision maker must select exactly one (without recall of preceding observations) and his objective is to minimize the expected rank E_n of the selected option. Here *it is* known that $\inf_\tau E_n(\tau)$ converges as $n \to \infty$, and that $1.908 < \lim_{n\to\infty} \inf_\tau E_n(\tau) \le 2.31$ where the lower bound can be somewhat improved. From the probabilistic point of view it would be most interesting to be able to prove that the limit does not coincide with the upper bound which can be obtained by memoryless threshold strategies.

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