The Brownian Excursion multi-dimensional local time density *

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Abstract

Expressions for the multi-dimensional densities of Brownian excursion local time are derived by two different methods: A direct method based on Kac's formula for Brownian functionals and an indirect one based on a limit theorem for Galton-Watson trees.

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1 Introduction

Throughout this paper, the standard Brownian motion (BM) will be denoted by $x(t)$. Three other classical BM are the reflecting BM: $x^+(t) \equiv |x(t)|$, the absorbing BM, killed at $0 : x^-(t)$ and the absorbing, two-barriers $[0, \xi]$ BM, killed at 0 and $\xi : \tilde{x}(t, \xi)$.

Fix $t > 0$ and denote the last zero of x before t and the first zero of x after t by

$$G(t) := \sup\{s : x \leq t; x(s) = 0\}$$

and

$$D(t) := \inf\{s : s \geq t; x(s) = 0\}.$$

The processes restricted to $[G(t), t]$ and $[G(t), D(t)]$ are called the meandering process ending at $t : Z(u) := x^+(G(t) + u)$, $0 \leq u \leq L^-(t) := t - G(t)$ and the excursion process straddling $t : Y(u) := x^+(G(t) + u), 0 \leq u \leq L(t) := D(t) - G(t)$, respectively. The standard scaled excursion (BE) is $X(u) := [Y(u)|L = 1]$; note that $Y(u) \stackrel{d}{\equiv} \sqrt{\ell} X(u/\ell)$ when $L = \ell$. The distributions of G and L are well known: see Chung [2, Theorem 1], Levy [18, equ. (15) and (26)].

The local time of $x(t)$ at a, denoted by

$$t^+(t, a) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_0^t I_{[a, a+\varepsilon]}(x(s))\, ds,$$

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and the total local time of the standard scaled excursion X at a, denoted by $\tau^+(a)$, have been studied by several authors (note that for an excursion of length ℓ we have: $\tau^+(\ell,a) \stackrel{d}{\equiv} \sqrt{\ell}\tau^+(a/\sqrt{\ell})$. See for instance Getoor and Sharpe [8], Knight [16], Cohen and Hooghiemstra [3], Hooghiemstra [11], Drmota and Gittenberger [5].

Several representations of the one dimensional Brownian Excursion local time density are known. Results for the two dimensional local time density can be found in [3] and [5]. In both papers indirect methods have been used: Approaching via queuing theory and random trees, respectively.

In this paper we aim at deriving expressions for all dimensions and offer two different methods to do this: a direct and an indirect one. The first one is a direct computation by means of Kac's formula for Brownian functionals and a method of Chung and the second one is based on the fact that the process consisting of the – suitably normalized – generation sizes of a Galton-Watson branching process conditioned on the total progeny weakly converges to Brownian excursion local time. So in order to obtain density representations one has to count the number of trees given the tree size and the sizes of the layers corresponding to the local times under consideration and to let the tree size tend to ∞ afterwards. In this ways we obtain representations for the multi-dimensional density which generalize Cohen and Hooghiemstra's [3, 11] result for the one dimensional density $f_x(y)$ of $\tau^+(x)$ given by

$$f_x(y) = \frac{1}{i\sqrt{2\pi}} \int_S \frac{\alpha e^{\alpha}}{\sinh^2(x\sqrt{\alpha}\,)} \exp\left(-\frac{y}{\sqrt{2}} \frac{\sqrt{\alpha} e^{x\sqrt{2\alpha}}}{\sinh(x\sqrt{\alpha}\,)} \right) d\alpha \tag{1}$$

where $S := (a - i\infty, a + i\infty)$, $a > 0$, is a straight line parallel to the imaginary axis.

1.1 Applications of Brownian excursion and its local time

1.1.1 $M/G/1$ queues

Applications of the BE are numerous: For instance, consider a $M/G/1$ queueing system. There the customers arrive according to a Poisson process $(\pi_t, t \geq 0)$ with rate α^{-1} where $\alpha > 0$. Denote the arriving time of the n-th customer by t_n and the service time by s_n which is assumed to be independent of the arrival process π_t. Then the actual waiting time process is defined by $w_1 := 0$, $w_{n+1} := \max\{0, w_n + s_n - (t_{n+1} - t_n)\}$ and the virtual waiting time process by

$$v_t := \max\{0, w_{\pi_t} + s_{\pi_t} - (t - t_{\pi_t})\}, \quad t \geq 0.$$

Furthermore, denote the length of the first busy period by ℓ. Then Cohen and Hooghiemstra [3] have shown that for arbitrary $\delta > 0$ the following limit theorem holds:

$$\left(\left(\frac{v_{su}}{\sqrt{2\alpha s}} \middle| s < \ell \leq s + \delta \right), 0 \leq u \leq 1 \right) \xrightarrow{d} X(u), \qquad s \to \infty.$$

In this context the BE local time process appears as the weak limit of the (suitably normalized) number of downcrossings of the virtual waiting time process, i.e.

$$d(v) = \#\{t : 0 \leq t \leq \ell, v_t = v\},$$

($\#A$ denotes the cardinality of A) conditioned on the number of customers served during the first busy period (see [3, Sec. 7]).

1.1.2 Modeling branching processes, random trees and relations to theoretical computer science

Another BE application is the number of nodes at some level in a random tree. Consider a simply generated random tree (according to the notion of Meir and Moon [22]) or, equivalently, the family tree of a Galton-Watson branching process conditioned on the total progeny. Then BE appears as the weak limit of the contour process of this tree, i.e. the process constructed of the distances of the nodes from the root when traversing the tree (for details see [1] or [9]). The generation sizes of such branching processes converge weakly to BE local time. Since we will use the correspondence between branching processes and local time later (cf. Section 2.3) we do not go into details now.

Note that binary trees also belong to the class of simply generated trees and that recently Gutjahr and Pflug [10] showed that the process constructed of the leaf heights during tree traversal converges to BE. The corresponding "local" result can be found in [5]. Binary trees play an important role in theoretical computer science (see e.g. [17]). Consider, e.g., searching data stored in a binary tree by level order traversal (see [14, pp.82]) of the tree. Then the stack size process (analyzed in [13]) can be described by the leaf height process. Thus the number of downcrossings of the stack size process can be described by BE local time.

Furthermore, it should be mentioned that not only nodes or leaves at some level in a random tree but also nodes of an arbitrarily given degree yield BE local time as limit process (see [4]). We also want to mention that BE is used as model of continuum trees, a concept introduced by Aldous [1]. Readers interested in this area should for instance consult [7].

1.1.3 Dynamical algorithms

The priority queue in Knuth's model is combinatorially equivalent to a Markov Stack, which is asymptotically equivalent to a BE (see Louchard et al [21]). So the distribution of the size of this structure is asymptotically given, on $[0,1]$, by the BE local time.

For other applications of Kac's formula see Louchard [19]. Let us mention that Brownian motion (and more general Gaussian processes) has many applications in data structures and algorithms analysis. See for instance Louchard [20] where other references can be found.

This paper is organized as follows. In Sec. 2 we summarize basic notations and known results. Some preliminary formulas based on Kac's formula are given in Sec. 3 and how to invert these formulas is shown in Sec. 4. Sec. 5 is devoted to the general multi-dimensional density. Starting from dimension 2 and 3, we proceed to dimension $d > 3$, with two distinct proofs. Sec. 6 deals with the moments and Sec. 7 concludes the paper. An Appendix collects some technical results we need in the text.

We would like to mention that MAPLE was of great help in computing some complicated expressions (with some guidance of course).

2 Basic notations and known results

2.1 Chung's method

Denote by $\eta(t)$ any of the processes defined in the previous section. Then we will use the notation

$$E_a[B(\eta)] := \Pr[B|\eta(0) = a]$$

where $B(\eta)$ is an event belonging to the Borel field generated by $\eta(t)$. Furthermore denote by $\mathcal{L}_\alpha(f(x)) := \int_0^\infty e^{-\alpha x} f(x)\, dx$ the Laplace transform of $f(x)$ and due to the frequent occurrence of $\sqrt{2\alpha}$ in the sequel we set for simplicity of notation $\sqrt{\cdot} := \sqrt{2\alpha}$.

Then the classical density (for $\eta(t) = x(t)$)

$$p(t,x,y)dy := E_x[x(t) \in dy] = \frac{1}{\sqrt{2\pi t}} \exp\left[-\frac{(x-y)^2}{2t}\right] dy$$

implies

$$\mathcal{L}_\alpha(p(t,x,y)) = \frac{\exp(-\sqrt{\cdot}|x-y|)}{\sqrt{\cdot}} \tag{2}$$

where the Laplace transform is taken with respect to t.

The maxima of the meander and excursion processes have been studied by Chung [2], Equations (3.17), (4.9), p. 169 and 177; and Kennedy [15, Theorem 1]. They obtain

$$E_0\left[\max_{0\le u\le r} Z(u) \le \xi | L^-(t) = r\right] = f_1(\xi^2/2r)$$

where

$$f_1(x) = \sum_{n=-\infty}^{+\infty} (-1)^n \exp(-n^2 x)$$

and

$$E_0\left[\max_{0\le u\le \ell} Y(u) \le \xi | L(t) = \ell\right] = f_2(2\xi^2/\ell)$$

where

$$f_2(x) = 1 + 2\sum_{n=1}^{\infty} (1-2n^2 x)\exp(-n^2 x).$$

The distributions of the maxima of Z and Y are particular cases of the following observation by Chung (see [2, p. 163]). Let $B = B(x^-)$ be an event belonging to the Borel field generated by x^- and let $\Pi(t,x,y)dy := E_x(B, x^-(t) \in dy)$, $x, y > 0$. The same event for Brownian meander Z and Brownian excursion Y has the following probability:

$$\begin{aligned} Pr[B(Z), Z(r) \in dy | L^-(t) = r] &= \sqrt{2\pi r}\frac{1}{2}\Pi_x(r,0,y)dy \\ Pr[B(Y)|L(t) = \ell] &= \sqrt{8\pi\ell^3}\frac{1}{4}\Pi_{x,y}(\ell,0,0) \end{aligned} \tag{3}$$

where the subscripts x and y denote partial derivatives of $\Pi(t,x,y)$.

Example: Set

$$\varphi(t,x,y,\xi)\, dy := E_x[\tilde{x}(t,\xi) \in dy] = E_x[B(x^-), x^-(t) \in dy]$$

where $B(x^-) = [\max_{0\le s\le t} x^-(s) < \xi]$. The analogous events for Z and Y are

$$B(Z) = \left[\max_{0\le s\le r} Z(s) < \xi\right] \qquad \text{and} \qquad B(Y) = \left[\max_{0\le s\le l} Y(s) < \xi\right].$$

Writing hitting times as $m_a(\eta) := \inf\{s : \eta(s) = a\}$ for any of the considered processes (m_a without specifying a process refers to the standard BM $x(t)$) we have by (3)

$$f_1(\xi^2/2t)/\sqrt{2\pi t} = \frac{1}{2}\left[\frac{\partial}{\partial x}E_x[m_\xi(x^-) > t]\right]_{x=0} = \frac{1}{2}\int_0^\xi \varphi_x(t,0,y,\xi)\,dy$$

$$f_2(2\xi^2/t)/\sqrt{8\pi t^3} = \frac{1}{4}\varphi_{x,y}(t,0,0,\xi).$$

2.2 Kac's formula for Brownian functionals

Let $h \geq 0$ be a piecewise continuous function and let $\mathcal{G}$ be the differential operator

$$(\mathcal{G}u)(a) := \frac{1}{2}u''(a) - h(a)u(a).$$

Kac's formula states that, for $\alpha > 0$ and $f \in C(R^1)$.

$$u(a) = E_a \int_0^\infty e^{-\alpha t} \exp\left(-\int_0^t h[x(s)]\,ds\right) f(x(t))\,dt \tag{4}$$

is the bounded solution of

$$(\alpha - \mathcal{G})u = f. \tag{5}$$

The solution of (5) is given by $u(a) = \int G(a,b)f(b)\,db$ where the Green function G is given by

$$G(a,b) = G(b,a) = 2W^{-1}g_1(a)g_2(b), \qquad a \leq b,$$

where $0 < g_1 \in\uparrow, 0 < g_2 \in\downarrow$ are independent solutions of $\mathcal{G}g = \alpha g$ and W is their constant positive Wronskian: $W = g_1'g_2 - g_1g_2'$. (See Itô and McKean [12, par. 2.6])

If we add to h in (4) another function h^*, the modified Green function $G^*(a,b)$ satisfies the relation

$$G^*(a,b) = G(a,b) - \int G(a,x)h^*(x)G^*(x,b)\,dx$$

(see Itô and McKean [12, p. 67]).

Indeed, let w_t^+ be the shifted path and $k(t) := \int_0^t h(x(s))ds$, $k^*(t) := \int_0^t h^*(x(s))ds$, then

$$\begin{aligned}
Gf - u^* &= E.\left[\int_0^\infty e^{-\alpha t - k(t)}[1 - e^{-k^*(t)}]f(x(t))\,dt\right] \\
&= E.\left[\int_0^\infty e^{-\alpha t - k(t)}[\int_0^t e^{k^*(t-s,\omega_s^+)}k^*(ds)]f(x(t))\,dt\right] \\
&= E.\left[\int_0^\infty k^*(ds)\int_s^\infty e^{-\alpha t - k(t)}e^{-k^*(t-s,\omega_s^+)}f(x(t))\,dt\right] \\
&= E.\left[\int_0^\infty e^{-\alpha s - k(s)}k^*(ds)\int_0^\infty e^{-\alpha t - k(t,\omega_s^+)}e^{-k^*(t,\omega_s^+)}f(x(t,\omega_s^+))\,dt\right] \\
&= E.\left[\int_0^\infty e^{-\alpha s - k(s)}h^*(x(s))u^*(x(s))\,ds\right] \\
&= \int_{-\infty}^{+\infty} G(\cdot,x)h(x)\left[\int_{-\infty}^{+\infty} G^*(x,y)f(y)\,dy\right]dx
\end{aligned}$$

If, in particular, we take for $h^*(x) = \gamma I_{(\xi,\eta)}(x)/(\eta - \xi)$ (where $I_{(\xi,\eta)}$ is the indicator function of the interval (ξ, η) with $\eta > \xi$) and let $\eta \downarrow \xi$, the modified Green function is given by

$$G^*(a,b) = G(a,b) - \gamma \frac{G(a,\xi)G(\xi,b)}{1 + \gamma G(\xi,\xi)}. \tag{6}$$

This corresponds to adding $\gamma t^+(t,\xi)$ to $\int_0^t h[x(s)]\, ds$. Letting $\gamma \uparrow \infty$, we get from (6)

$$G(a,b) - \frac{G(a,\xi)G(\xi,b)}{G(\xi,\xi)} \tag{7}$$

which corresponds to

$$\int_0^\infty e^{-\alpha t} E_a \left[\exp\left[-\int_0^t h[x(s)]\, ds \right], t < m_\xi, x(t) \in db \right] dt \qquad a, b < \xi.$$

2.3 Random trees and local time

Consider a Galton-Watson branching process X with offspring distribution ξ given by

$$Pr[\xi = k] = \frac{\psi^k \varphi_k}{\varphi(\psi)}$$

where φ_k are non-negative numbers and $\varphi(t) = \sum_{k \geq 0} \varphi_k t^k$ is the generating function (GF) associated to X. ψ is the least positive solution of $\psi \varphi'(\psi) = \varphi(\psi)$ which is equivalent to the assumption that the branching process is critical. The variance of ξ is given by

$$\sigma^2 = \frac{\psi^2 \varphi''(\psi)}{\varphi(\psi)}. \tag{8}$$

Now let us introduce a process $(L_n(a); a \geq 0)$ defined in the following way: If a is an integer, then let $L_n(a)$ denote the generation size of the a-th generation of the process X conditioned on the total progeny to be n. For non-integer a define

$$L_n(a) = (\lfloor a \rfloor + 1 - a) L_n(\lfloor a \rfloor) + (a - \lfloor a \rfloor) L_n(\lfloor a \rfloor + 1), \quad a \geq 0.$$

In [5] the following theorem is proved:

Theorem 2.1 *Assume that $\varphi(t)$ has a positive or infinite radius of convergence R and suppose that the equation*

$$t \varphi'(t) = \varphi(t)$$

has a minimal positive solution $\tau < R$ and that σ^2 defined by (8) is finite. Then the following limit theorem holds:

$$\frac{1}{\sqrt{n}} L_n(a\sqrt{n}) \xrightarrow{w} \frac{\sigma}{2} \tau^+ \left(\frac{\sigma}{2} a \right)$$

in $C[0,\infty)$, as $n \to \infty$.

This theorem enables us to determine the multi–dimensional local time densities: The theorem holds for any class of trees and so we choose the class of planted plane trees where the corresponding GF has the simple shape $\varphi(t) = 1/(1-t)$. Let b_n be the number of planted plane trees consisting of n nodes. Furthermore denote by $b_{k_1\cdots k_d n}^{(r_1\cdots r_d)}$ the number of trees of size n having k_i nodes in layer r_i, where $r_1 < \ldots < r_d$. Then by setting

$$\begin{aligned} b_0(z,u) &= u \\ b_{i+1}(z,u) &= \frac{z}{1-b_i(z,u)} \end{aligned} \tag{9}$$

and

$$a(z) = \sum_{n\geq 0} b_n z^n = \frac{1-\sqrt{1-4z}}{2}$$

and using standard combinatorial techniques (readers not familiar with these techniques may consult e.g. [6]) we can write down the generating function of these numbers in the form

$$\begin{aligned} g(z,u_1,\ldots,u_d) &= \sum_{k_1,\ldots,k_d,n\geq 0} b_{k_1\cdots k_d n}^{(r_1\cdots r_d)} u_1^{k_1}\cdots u_d^{k_d} z^n \\ &= b_{r_1}(z, u_1 b_{h_{12}}(z, u_2 b_{h_{23}}(\cdots u_{d-1} b_{h_{d-1,d}}(z, u_d a(z))\cdots)) \end{aligned}$$

where $h_{ij} = r_j - r_i$.

The multi-dimensional density can be determined by evaluating the proper coefficient of $g(z,u_1,\ldots,u_d)$. If we set $r_i = \lfloor \rho_i\sqrt{n}\rfloor$ and $k_i = \lfloor y_i\sqrt{n}\rfloor$ and denote by $f_{\rho_1\cdots\rho_d}(y_1,\ldots,y_d)$ the joint density of $\tau^+(\rho_1),\ldots,\tau^+(\rho_d)$, then by Theorem 2.1 we have after normalizing according to the fact that $\sigma^2 = 2$ for $\varphi(t) = 1/(1-t)$

$$f_{\rho_1/\sqrt{2},\ldots,\rho_d/\sqrt{2}}\left(y_1\sqrt{2},\ldots,y_d\sqrt{2}\right) = 2^{-d/2}\lim_{n\to\infty}\frac{n^{d/2}}{b_n}[z^n u_1^{k_1}\cdots u_d^{k_d}]g(z,u_1,\ldots,u_d) \tag{10}$$

where the symbol $[z^n]f(z)$ denotes the coefficient of z^n in the power series expansion of $f(z)$.

3 Preliminary formulas

In this section, we define some auxiliary functions built on Kac's formula and Chung's method. We apply these results to dimension 1. Throughout the whole paper assume that (ρ_i) is a strictly monotonically increasing sequence of non-negative real numbers. Let

$$\varphi_d(\alpha) := \int_0^\infty [e^{-\alpha t}-1]E\left[e^{-\beta_1\sqrt{t}\tau^+(\rho_1/\sqrt{t})-\cdots-\beta_d\sqrt{t}\tau^+(\rho_d/\sqrt{t})}\right]\frac{dt}{\sqrt{2\pi t^3}}$$

and

$$\bar\varphi_d(\alpha) := \int_0^\infty [e^{-\alpha t}-1]E\left[e^{-\beta_1\sqrt{t}\tau^+(\rho_1/\sqrt{t})-\cdots-\beta_{d-1}\sqrt{t}\tau^+(\rho_{d-1}/\sqrt{t})}[t < m_{\rho_d}]\right]\frac{dt}{\sqrt{2\pi t^3}}.$$

where $m_.$ is related to the Brownian Excursion of duration t.

Lemma 1 *We have*

$$\varphi_d(\alpha) = \Psi_d(\alpha) - \Psi_d(0) \qquad \textit{and} \qquad \bar\varphi_d(\alpha) = \bar\Psi_d(\alpha) - \bar\Psi_d(0) \tag{11}$$

with

$$\Psi_d(\alpha) = \frac{1}{2}\frac{\partial}{\partial a}\frac{\partial}{\partial b} G_d^{**}(a,b)|_{a=b=0} \tag{12}$$

and

$$\bar{\Psi}_d(\alpha) := \frac{1}{2}\frac{\partial}{\partial a}\frac{\partial}{\partial b} \bar{G}_d^{**}(a,b)|_{a=b=0} \tag{13}$$

where

$$G_d^{**}(a,b) = \begin{cases} G^*(a,b) - \beta_1 \dfrac{G^*(a,\rho_1)G^*(\rho_1,b)}{1+\beta_1 G^*(\rho_1,\rho_1)} & \textit{for } d=1 \\ G_{d-1}^{**}(a,b) - \beta_d \dfrac{G_{d-1}^{**}(a,\rho_d)G_{d-1}^{**}(\rho_d,b)}{1+\beta_d G_{d-1}^{**}(\rho_d,\rho_d)} & \textit{for } d>1 \end{cases} \tag{14}$$

and

$$\bar{G}_d^{**}(a,b) = \begin{cases} G^*(a,b) - \dfrac{G^*(a,\rho_1)G^*(\rho_1,b)}{G^*(\rho_1,\rho_1)} & \textit{for } d=1 \\ G_{d-1}^{**}(a,b) - \dfrac{G_{d-1}^{**}(a,\rho_d)G_{d-1}^{**}(\rho_d,b)}{G_{d-1}^{**}(\rho_d,\rho_d)} & \textit{for } d>1 \end{cases} \tag{15}$$

with

$$G^*(a,b) = G(a,b) - \frac{G(a,0)G(0,b)}{G(0,0)}$$

and

$$G(a,b) = \frac{\exp\left(-\sqrt{\cdot}|a-b|\right)}{\sqrt{\cdot}}.$$

Proof: By (2) we have

$$G(a,b)\,db = \int_0^\infty e^{-\alpha t} E_a[x(t) \in db]\,dt.$$

Inserting $h \equiv 0$ and $f(y) = I_{db}(y)$ into Kac's formula and adding $\gamma t^+(t,0)$ as described before formula (7) yields

$$\begin{aligned} G^*(a,b)\,db &= \lim_{\gamma\uparrow\infty} \int_0^\infty e^{-\alpha t} E_a[\exp(-\gamma t^+(t,0)), x(t) \in db]\,dt \\ &= \int_0^\infty e^{-\alpha t} E_a[x(t) \in db, t < m_0]\,dt. \end{aligned}$$

Adding a local time $\beta_1 t^+(t,\rho_1)$ to the exponent gives a modification of the Green function according to (6) and thus we have

$$G_1^{**}(a,b) = \int_0^\infty e^{-\alpha t} E_a\left[e^{-\beta_1 t^+(t,\rho_1)} x(t) \in db, t < m_0\right]\,dt$$

and continuing in this way yields the recursion (14). For obtaining (15) we have to take into account that in the d-th step we restrict to $[t < m_{\rho_d}]$ (in the way we got (6) and (7), i.e. by adding $\gamma t^+(t,\rho_d)$ to the exponent and letting $\gamma \uparrow \infty$) instead of adding the local time $\beta_1 t^+(t,\rho_1)$.

Finally, noting that we deal with $Y(s)$ conditioned to have length t instead of $x^-(t)$ we have to apply Chung's result (second formula of (3)) which yields (11) as desired.

□

Example: For instance, when $d = 1$ we get

$$\Psi_1(\alpha) = -\sqrt{\cdot}\frac{\sqrt{\cdot} + \beta_1(1 + e^{-2\sqrt{\cdot}\rho_1})}{\sqrt{\cdot} + \beta_1(1 - e^{-2\sqrt{\cdot}\rho_1})} \tag{16}$$

in concordance with Louchard [19, Sec. 5.1]. Note that $\lim_{\beta_1 \to 0} \Psi_1(\alpha) = -\sqrt{\cdot}$

Application to dimension 1

Let us recover Hooghiemstra [11, (2.3)]. We have with $S := [a - i\infty, a + i\infty], a > 0$

$$\begin{aligned}
&\int_0^\infty e^{-\alpha t} E[e^{-\beta_1\sqrt{t}\tau^+(\rho_1/\sqrt{t})}, t > m_{\rho_1}]\frac{dt}{\sqrt{2\pi t^3}} \\
&= \frac{1}{4i\sqrt{2\pi}}\int_0^\infty e^{-\alpha t}\frac{dt}{\sqrt{2\pi t^3}}\int_{S\cdot t} e^{z/2}dz \int_0^\infty dy \frac{e^{-\beta_1\sqrt{t}y}z}{\sinh^2(\rho_1\sqrt{z}/\sqrt{t})}\exp\left[\frac{-y\sqrt{z}e^{\rho_1\sqrt{z}/\sqrt{t}}}{2\sinh(\rho_1\sqrt{z}/\sqrt{t})}\right] \\
&= \frac{1}{4i\sqrt{2\pi}}\int_0^\infty e^{-\alpha t}\frac{dt}{\sqrt{2\pi t^3}}\int_{S\cdot t}\frac{e^{z/2}z\,dz}{\sinh^2(\rho_1\sqrt{z}/\sqrt{t})}\quad\frac{1}{\beta_1\sqrt{t} + \frac{\sqrt{z}}{2}\frac{e^{\rho_1\sqrt{z}/\sqrt{t}}}{\sinh(\rho_1\sqrt{z}/\sqrt{t})}}
\end{aligned}$$

(Interchange of integrations can be justified). Set $u = z/t$, we derive

$$\begin{aligned}
&\frac{1}{4i\sqrt{2\pi}}\int_0^\infty e^{-\alpha t}\frac{dt}{\sqrt{2\pi t^3}}\int_S\frac{e^{ut/2}\ ut\ t\,du}{\sqrt{t}\ \sinh(\rho_1\sqrt{u})[\beta_1\sinh(\rho_1\sqrt{u}) + \frac{\sqrt{u}}{2}e^{\sqrt{u}\rho_1}]} \\
&\quad = \frac{1}{4i\sqrt{2\pi}}\frac{1}{\sqrt{2\pi}}\int_S\frac{u\ du}{(\alpha - u/2)\sinh(\rho_1\sqrt{u})[\beta_1\sinh(\rho_1\sqrt{u}) + \frac{\sqrt{u}}{2}e^{\rho_1\sqrt{u}}]}
\end{aligned}$$

and finally, set $u = 2w$. We obtain:

$$\frac{1}{2\pi i}\int_S\frac{2w\ dw}{(\alpha - w)\sinh(\rho_1\sqrt{2w})[2\beta_1\sinh(\rho_1\sqrt{2w}) + \sqrt{2w}e^{\rho_1\sqrt{2w}}]}$$

and by residues,

$$= \frac{2\alpha}{\sinh(\rho_1\sqrt{\cdot})[2\beta_1\sinh(\rho_1\sqrt{\cdot}) + \sqrt{\cdot}e^{\rho_1\sqrt{\cdot}}]}. \tag{17}$$

This must be equivalent to $\Psi_1(\alpha) - \bar{\Psi}_1(\alpha)$ which is indeed the case.

4 Inverting the formulas

If we want to go the other way, it is easy to check that any result of the form

$$\int_0^\infty [e^{-\alpha t} - 1]E[e^{-\beta_1\sqrt{t}\tau^+(\rho_1/\sqrt{t})\cdots-\beta_d\sqrt{t}\tau^+(\rho_d/\sqrt{t})}, \mathcal{A}]\frac{dt}{\sqrt{2\pi t^3}} = \Xi_d(\alpha) - \Xi_d(0) \tag{18}$$

where $\mathcal{A} = I$ or $\mathcal{A} = t < m_{\rho_d}$ and $\Xi_d = \Psi_d$ or $\Xi_d = \bar{\Psi}_d$, respectively, leads to the following forms.

Lemma 2 *Set* $\mathcal{B}_d(t) := e^{-\beta_1\sqrt{t}\tau^+(\rho_1/\sqrt{t})\cdots-\beta_d\sqrt{t}\tau^+(\rho_d/\sqrt{t})}$. *We obtain the following inversion formulas:*

$$\begin{aligned}
E[\mathcal{B}_d(1)]-1 &= \frac{1}{\sqrt{2\pi i}}\int_S e^u[\Psi_d(u)+\sqrt{2u}]\,du \\
E[\mathcal{B}_{d-1}(1), t<1]-1 &= \frac{1}{\sqrt{2\pi i}}\int_S e^u[\bar{\Psi}_d(u)+\sqrt{2u}]\,du \\
E[\mathcal{B}_d(1), 1>m_{\rho_d}] &= \frac{1}{\sqrt{2\pi i}}\int_S e^u[\Psi_d(u)-\bar{\Psi}_d(u)]\,du
\end{aligned}$$

Proof: By Lemma 1 we know that

$$\int_0^\infty e^{-\alpha t}E[\mathcal{B}_d(t), t>m_{\rho_d}]\frac{dt}{\sqrt{2\pi t^3}} = \Psi_d(\alpha)-\bar{\Psi}_d(\alpha)$$

(we have no singularity at $t=0$). In particular,

$$\int_0^\infty e^{-\alpha t}\Pr[t>m_{\rho_1}]\frac{dt}{\sqrt{2\pi t^3}} = -\sqrt{\cdot}-\bar{\Psi}_1(\alpha).$$

Also, we know that $\int_0^\infty[e^{-\alpha t}-1]\frac{dt}{\sqrt{2\pi t^3}} = -\sqrt{\cdot}$. Differentiation of $\varphi_d(\alpha)$ and $\bar{\varphi}_d(\alpha)$ w.r.t. α leads to $\mathcal{L}_\alpha[\mathcal{B}_d(t)-1] = \sqrt{\cdot}+\Psi_d(\alpha)$ and similarly for $\bar{\Psi}_d(\alpha)$. Hence we obtain the above inversion formulas. □

5 Multi-dimensional densities

In this section, we first analyze dimension 2 and 3. Then we compute the general form of the transforms and invert them to get the densities.

We offer two different proofs of our results: the first one is based on some properties of G^{**}. The other one is based on Cauchy's formula applied to (10) and singularity analysis.

5.1 Dimension 2

Let us first consider the 2-dimensional density.

Set

$$\begin{aligned}
M &:= \sup_{u\in[0,1]}[Y(u)|L=1] \\
M_t &:= \sup_{u\in[0,t]}[Y(u)|L=t]
\end{aligned}$$

Then, we have the following decomposition

$$\begin{aligned}
E[e^{-\beta_1\tau^+(\rho_1)-\beta_2\tau^+(\rho_2)}] &= E[e^{-\beta_1\tau^+(\rho_1)-\beta_2\tau^+(\rho_2)}, M>\rho_2] \\
&+E[e^{-\beta_1\tau^+(\rho_1)}, \rho_1<M<\rho_2]+P[M<\rho_1]
\end{aligned}$$

and with our transforms, this corresponds to $\Psi_2(\alpha) = (\Psi_2(\alpha)-\bar{\Psi}_2(\alpha))+(\bar{\Psi}_2(\alpha)-\bar{\Psi}_1(\alpha))+\bar{\Psi}_1(\alpha)$.

Remark. We want to indicate the correspondence of our transforms to some known results:

i) For instance, in Louchard [19, p. 494]) we have

$$\int_0^\infty (e^{-\alpha t}-1)\,\Pr[M_t<\xi/\sqrt{t}]\frac{dt}{\sqrt{2\pi t^3}}=\frac{1-\sqrt{\cdot}\xi\coth(\sqrt{\cdot}\xi)}{\xi}$$

which can be reproved using

$$\bar{\Psi}_1(\alpha)=-\sqrt{\cdot}\coth(\sqrt{\cdot}\rho_1).$$

ii) Also Hooghiemstra's one dimensional density [11, (2.3)] corresponds to $\Psi_1(\alpha)-\bar{\Psi}_1(\alpha)$ (which has been worked out at the end of Sec. 3).

iii) The Laplace transform of the two dimensional local time density as mentioned in Cohen and Hooghiemstra [3, Theorem 6.3] can be re-obtained by inverting $\Psi_2(\alpha)-\bar{\Psi}_1(\alpha)$. We have checked (with MAPLE) that this indeed leads to [3, (6.11)].

iv) From now on let $Sh_i := \sinh(\sqrt{\cdot}\rho_i), Sh_{ij} := \sinh[\sqrt{\cdot}(\rho_i-\rho_j)]$, and similarly for Ch_i, Ch_{ij}. Then $\bar{\Psi}_2(\alpha)-\bar{\Psi}_1(\alpha)$ is given by

$$\frac{2\alpha Sh_{21}}{[\sqrt{\cdot}Sh_2+2\beta_1 Sh_1 Sh_{21}]Sh_1}$$

and inverting this leads to [5, Theorem 7.1(second term)]:

$$\int_S \frac{e^\alpha \alpha}{\sqrt{2\pi}iSh_1^2}e^{-y_1\sqrt{\frac{\alpha}{2}}\frac{Sh_2}{Sh_1Sh_{21}}}d\alpha$$

However, we need $\Psi_2(\alpha)-\bar{\Psi}_2(\alpha)$ to get the transforms of the density. With the help of MAPLE we obtain by Lemma 1 (we do not write down computational details here, since they are covered by the proof of general dimensions which is deferred to Section 5.4)

$$E[e^{-\beta_1\tau^+(\rho_1)-\beta_2\tau^+(\rho_2)},1>m_{\rho_2}]=\frac{1}{\sqrt{2\pi}i}\int_S e^\alpha\frac{2\alpha^2\,d\alpha}{f_3^2[2\beta_2+\sqrt{\cdot}+f_4/f_3]} \tag{19}$$

with

$$\begin{aligned} f_3 &:= \beta_1\sqrt{2}Sh_1Sh_{21}+\sqrt{\alpha}Ch_1Sh_{21}+\sqrt{\alpha}Sh_1Ch_{21}\\ f_4 &:= \sqrt{\cdot}(\sqrt{\alpha}Sh_1Sh_{21}+\beta_1\sqrt{2}Sh_1Ch_{21}+\sqrt{\alpha}Ch_1Ch_{21}) \end{aligned}$$

Let us first invert (19) on β_2. This leads to

$$\frac{e^\alpha\alpha^2}{f_3^2}e^{-y_2[\frac{\sqrt{\cdot}}{2}+\frac{f_4}{2f_3}]}.$$

Now write f_3 as $\sqrt{2}Sh_1Sh_{21}[\beta_1+C_1]$ with

$$C_1=\frac{\sqrt{\alpha}[Ch_1Sh_{21}+Sh_1Ch_{21}]}{\sqrt{2}Sh_1Sh_{21}}=\frac{\sqrt{\alpha}Sh_2}{\sqrt{2}Sh_1Sh_{21}}$$

By standard arguments, this leads to the density

$$\frac{1}{\sqrt{2\pi}i}\int_S\frac{e^\alpha\alpha^2e^{-C_1y_1}}{2Sh_1^2Sh_{21}^2\beta_1^2}e^{-\frac{y_2\sqrt{\cdot}}{2}-\frac{y_2}{2}\frac{f_4-\sqrt{\cdot}\sqrt{2}Sh_1Ch_{21}C_1}{\sqrt{2}Sh_1Sh_{21}\beta_1}}\,d\alpha.$$

The second exponential term gives

$$\begin{aligned}
&\exp\left[\frac{-y_2\sqrt{\cdot}E^{2,1}}{2Sh_{21}} - \frac{y_2\sqrt{\cdot}\sqrt{\alpha}}{2\sqrt{2}Sh_1Sh_{21}\beta_1}[Sh_1Sh_{21} - \frac{Ch_{21}^2Sh_1}{Sh_{21}}]\right] \\
&= \exp\left[-y_2\frac{\sqrt{\cdot}E^{2,1}}{2Sh_{21}} + y_2\sqrt{\cdot}\sqrt{\alpha}/[2\sqrt{2}Sh_{21}^2\beta_1]\right]
\end{aligned}$$

with $E^{i,j} := e^{\sqrt{\cdot}(\rho_i-\rho_j)}$, $E^k := e^{\sqrt{\cdot}\rho_k}$. But $\mathcal{L}_\beta[(\frac{x}{a})^{1/2}I_1(2(ax)^{1/2})] = e^{a/\beta}/\beta^2$ where I_1 is the first Bessel function and so we finally derive [5, Theorem 7.1].

5.2 Dimension 3

To get an idea of the general expression, let us analyze the dimension 3 density. MAPLE gives us

$$\Psi_3(\alpha) - \bar{\Psi}_3(\alpha) = \frac{\alpha^3}{2T_1^2[\beta_3 + \frac{\sqrt{\alpha}E^{3,2}}{\sqrt{2}Sh_{32}} + \frac{f_5}{T_1}]}$$

with $T_1 := \beta_2 D_2 + D_1$ and D_1, D_2 are linear in β_1. But the interesting fact is that: $\frac{f_5}{D_2} = -\frac{\alpha}{2Sh_{32}^2}$.

Proceeding as in Sec. 5.1 (we omit the details), we obtain the density

$$\begin{aligned}
&\frac{1}{\sqrt{2\pi i}}\int_S \frac{e^\alpha\alpha^3}{4Sh_1^2Sh_{21}^2Sh_{32}^2} e^{-\frac{y_3\sqrt{\alpha}E^{3,2}}{\sqrt{2}Sh_{32}} - y_2\frac{\sqrt{\alpha}Sh_{31}}{\sqrt{2}Sh_{21}Sh_{32}} - y_1\frac{\sqrt{\alpha}Sh_2}{\sqrt{2}Sh_1Sh_{21}}} \\
&\quad * \sqrt{\frac{y_2}{a_2}} I_1(2\sqrt{a_2y_2})\sqrt{\frac{y_1}{a_1}}I_1(2\sqrt{a_1y_1})\,d\alpha
\end{aligned}$$

with $a_2 := -\frac{\alpha y_3}{2Sh_{32}^2}, a_1 := -\frac{\alpha y_2}{2Sh_{21}^2}$.

5.3 Dimension d

It is now time to guess general expressions. We first conjecture the following Lemma for dimension d (the proof is given in Sec. 5.4).

Lemma 3 *We have*

A. $\Psi_d(\alpha) - \bar{\Psi}_d(\alpha) = \Theta(d)$ *with*

$$\Theta(d) := \frac{\alpha^d}{2[F_1(d)]^2[\beta_d + C_1(d) + C_2(d)D_2(d)/F_1(d)]}$$

where

$$\begin{aligned}
C_1(d) &= \sqrt{\frac{\alpha}{2}}E^{d,d-1}/Sh_{d,d-1} \\
C_2(d) &= -\frac{\alpha}{2Sh_{d,d-1}^2} \\
C_3(d) &= \sqrt{\frac{\alpha}{2}}\frac{Sh_{d,d-2}}{Sh_{d,d-1}Sh_{d-1,d-2}} \\
F_1(d) &= \beta_{d-1}D_2(d) + D_1(d)
\end{aligned}$$

$$\begin{aligned}
D_2(d) &= (\beta_{d-2}D_2(d-1)+D_1(d-1))\sqrt{2}Sh_{d,d-1} \\
&= \sqrt{2}Sh_{d,d-1}F_1(d-1) \\
D_1(d) &= C_3(d)D_2(d)+C_2(d-1)D_2(d-1)\sqrt{2}Sh_{d,d-1} \\
&= \sqrt{2}Sh_{d,d-1}C_3(d)F_1(d-1)+2Sh_{d,d-1}Sh_{d-1,d-2}C_2(d-1)F_1(d-2)
\end{aligned}$$

The coefficient of $\beta_1\cdots\beta_{d-2}$ in $D_2(d)$ equals $2^{(d-2)/2}Sh_1\prod_{l=2}^{d}Sh_{l,l-1}$ or, equivalently, the initial values of $F_1(d)$ are given by

$$F_1(1)=\frac{1}{\sqrt{2}}Sh_1 \qquad F_1(2)=Sh_1Sh_{21}(\beta_1+C_3(2))$$

B.

$$\bar{\Psi}_d(\alpha)-\bar{\Psi}_{d-1}(\alpha)=\Theta(d-1)-\frac{\alpha^d}{2[F_1(d)]^2[C_1(d)+C_2(d)D_2(d)/F_1(d)]}$$

Once we have proved Lemma 5.1, it is now routine to derive the following Theorem: Part A is computed as in Sec. 5.1, in Part B, we use the transforms

$$\mathcal{L}_\alpha\left[\frac{1}{a}(1-e^{ax})\right]=\frac{1}{\alpha(\alpha+a)}$$

Theorem 5.1 .

A. The d-dimensional density is given by

$$
\begin{aligned}
& f_{\rho_1\cdots\rho_d}(y_1, y_2, \ldots, y_d; M > \rho_d) = \\
& \quad \frac{1}{i\sqrt{2\pi}} \int_S \frac{e^\alpha \alpha^d}{2^{d-1} Sh_1^2 \prod_{l=2}^d Sh_{l,l-1}^2} \cdot \\
& \quad \cdot \exp\left[-y_d C_1(d) - \sum_{l=2}^{d-1} y_l C_3(l+1) - y_1 \frac{\sqrt{\alpha} Sh_2}{\sqrt{2} Sh_1 Sh_{21}}\right] \cdot \\
& \quad \cdot \prod_{l=1}^{d-1} \left[\sqrt{\frac{y_l}{a_l y_{l+1}}} I_1[2\sqrt{a_l y_l y_{l+1}}]\right] d\alpha
\end{aligned} \tag{20}
$$

with $a_l := -\frac{\alpha}{2Sh_{l+1,l}^2}$. *For* $d = 1$ *this coincides with (1).*

B. The constraint densities $f_{\rho_1\cdots\rho_d}(y_1, \ldots, y_k; \rho_{k+1} > M > \rho_k)$ *are given by (20) where we replace* d *by* k *and* $-y_k C_1(k)$ *by* $-y_k C_3(k+1)$.

5.4 The proofs

Of course, once we have a conjecture for the expressions, proofs are easier to develop.

5.4.1 Using some properties of G_d

Proof: *Proof of Lemma 5.1, Part A* Actually, we will use auxiliary functions $D_3(d)$, $D_4(d)$, and prove the following additional relations

$$
\begin{aligned}
D_2(d) &= \beta_{d-2} D_4(d) + D_3(d) \\
\frac{D_1(d)}{D_2(d)} &= C_3(d) + C_4(d) D_4(d)/D_2(d) \\
C_4(d) &= C_2(d-1) \\
\frac{D_3(d)}{D_4(d)} &= \frac{D_1(d-1)}{D_2(d-1)} \\
D_4(d) &= D_2(d-1)\sqrt{2} Sh_{d,d-1}
\end{aligned}
$$

The coefficient of $\beta_1 \cdots \beta_{d-1}$ in $D_4(d) = 2^{(d-2)/2} Sh_1 \prod_{l=2}^d Sh_{l,l-1}$.

We must analyze $G_d^{**}(a,b) - \bar{G}_d^{**}(a,b)$. With (14) and (15), this leads to

$$
\begin{aligned}
& \frac{G_{d-1}^{**}(a,\rho_d) G_{d-1}^{**}(\rho_d, b)}{[1 + \beta_d G_{d-1}^{**}(\rho_d,\rho_d)] G_{d-1}^{**}(\rho_d,\rho_d)} \\
& \quad = \frac{G_{d-1}^{**}(a,\rho_d) G_{d-1}^{**}(\rho_d, b)}{[G_{d-1}^{**}(\rho_d,\rho_d)]^2 [\beta_d + \frac{1}{G_{d-1}^{**}(\rho_d,\rho_d)}]}
\end{aligned} \tag{21}
$$

The proof of the conjecture is now divided into 2 parts: first we start backwards from the conjecture and see what this entails for our expressions. The purpose of going backwards is twofold.

a) From a methodological point of view, that's the way we could arrive at Th. 5.1, which was not evident from the examples.

b) From an algebraic point of view, it is the only way to derive all auxiliary functions and, more important, their mutual relations.

Then we proceed forwards from (21) ($d = 2$) to our conjecture by induction from $d-1$ to d (this is easy, we will not detail this part).

1. *Starting backwards:*

   a. We must have

$$G^{**}_{d-1}(a,\rho_d)G^{**}_{d-1}(\rho_d,b) = \frac{\alpha^{d-1}Sh_a Sh_b}{2H^2(d)} \tag{22}$$

for some $H(d)$ and (21) leads to

$$\frac{\alpha^{d-1}Sh_a Sh_b}{2[H(d)G^{**}_{d-1}(\rho_d,\rho_d)]^2[\beta_d + \frac{1}{G^{**}_{d-1}(\rho_d,\rho_d)}]} \tag{23}$$

Set $F_1(d) := H(d)G^{**}_{d-1}(\rho_d,\rho_d)$.
In the following, $C.$ denotes functions of $x.$ and $D.$, $f.$ denote functions of $x.$ and $\beta., D.(d)$ depends on $\beta_{d-2}, \beta_{d-3}, \cdots$
We must have

$$\frac{1}{G^{**}_{d-1}(\rho_d,\rho_d)} = C_1(d) + \frac{f_6(d)}{F_1(d)} = C_1(d) + \frac{f_6(d)}{H(d)G^{**}_{d-1}(\rho_d,\rho_d)}$$

With $f_6(d) = C_2(d)D_2(d)$
and $F_1(d) = \beta_{d-1}D_2(d) + D_1(d) = D_2(d)[\beta_{d-1} + D_1(d)/D_2(d)]$
We obtain

$$H(d) = \frac{f_6(d)}{1 - C_1(d)G^{**}_{d-1}(\rho_d,\rho_d)} \tag{24}$$

and

$$f_6(d)G^{**}_{d-1}(\rho_d,\rho_d) = [1 - C_1(d)G^{*}_{d-1}(\rho_d,\rho_d)][\beta_{d-1}D_2(d) + D_1(d)] \tag{25}$$

Now, from (14), $G^{**}_{d-1}(\rho_d,\rho_d) = [G^{**}_{d-2}(\rho_d,\rho_d) + \beta_{d-1}F_2(d)]/F_3(d)$ with

$$F_2(d) := [G^{**}_{d-2}(\rho_d,\rho_d)G^{**}_{d-2}(\rho_{d-1},\rho_{d-1}) - G^{**}_{d-2}(\rho_d,\rho_{d-1})^2]$$

and

$$F_3(d) := 1 + \beta_{d-1}G^{**}_{d-2}(\rho_{d-1},\rho_{d-1}).$$

(25) leads to

$$\begin{aligned} C_1(d) &= G^{**}_{d-2}(\rho_{d-1},\rho_{d-1})/F_2(d) && (26)\\ C_2(d) &= [1 - C_1(d)G^{**}_{d-2}(\rho_d,\rho_d)]/F_2(d) = -G^{**}_{d-2}(\rho_d,\rho_{d-1})^2/F_2^2(d) && (27)\\ \frac{D_1(d)}{D_2(d)} &= G^{**}_{d-2}(\rho_d,\rho_d)/F_2(d) && (28) \end{aligned}$$

By $(A \cdot 1)$ and $(A \cdot 3)$ with $l = d-1, m = d-2$, we derive

$$C_1(d) = \sqrt{\frac{\alpha}{2}} \frac{E^{d,d-1}}{Sh_{d,d-1}}, C_2(d) = -\frac{\alpha}{2Sh^2_{d,d-1}}, \tag{29}$$

which are the correct values.

b. Let us know turn to (22). Again, by (14), $G^{**}_{d-1}(a,\rho_d) = [G^{**}_{d-2}(a,\rho_d) + F_6]/F_3(d)$, with

$$F_6 = \beta_{d-1}[G^{**}_{d-2}(a,\rho_d)G^{**}_{d-2}(\rho_{d-1},\rho_{d-1}) - G^{**}_{d-2}(a,\rho_{d-1})G^{**}_{d-2}(\rho_{d-1},\rho_d)].$$

But by $(A \cdot 4)$ this leads simply to $G^{**}_{d-2}(a,\rho_d)/F_3(d)$.
(24) and (22) give

$$G^{**}_{d-2}(a,\rho_d) = \frac{Sh_a \alpha^{(d-1)/2} F_2(d)}{\sqrt{2} D_2(d)}.$$

This allows the computation of D_2. Indeed, by successive application of $(A \cdot 4)$, we derive

$$G^{**}_{d-2}(a,\rho_d) = G_1(a,\rho_d)/[F_3(d-1)F_3(d-2)\cdots F_3(2)]$$

and $G_1(a,\rho_d) = Sh_a E^{-d}\sqrt{2/\alpha}$.
But

$$F_2(d) = \frac{G^{**}_{d-2}(\rho_{d-1},\rho_{d-1})}{C_1(d)} = \frac{[G^{**}_{d-3}(\rho_{d-1},\rho_{d-1}) + \beta_{d-2}F_2(d-1)]}{C_1(d)F_3(d-1)}$$

hence, with $F_4(d) := [F_3(d-2)\cdots F_3(2)\sqrt{\frac{\alpha}{2}}E^d]$, we obtain

$$\begin{aligned} D_2(d) &= F_2(d)F_3(d-1)\alpha^{(d-1)/2}F_4(d)/\sqrt{2} \\ &= \frac{\alpha^{(d-1)/2}}{C_1(d)\sqrt{2}}[G^{**}_{d-3}(\rho_{d-1},\rho_{d-1}) + \beta_{d-2}F_2(d-1)]F_4(d) \end{aligned} \tag{30}$$

So $D_2(d) = \beta_{d-2}D_4(d) + D_3(d)$ with

$$\begin{aligned} D_4(d) &:= \frac{\alpha^{(d-1)/2}}{\sqrt{2}C_1(d)}F_2(d-1)F_4(d) \\ D_3(d) &:= \frac{\alpha^{(d-1)/2}}{\sqrt{2}C_1(d)}G^{**}_{d-3}(\rho_{d-1},\rho_{d-1})F_4(d) \end{aligned}$$

and $\frac{D_3(d)}{D_4(d)} = \frac{G^{**}_{d-3}(\rho_{d-1},\rho_{d-1})}{F_2(d-1)} = \frac{D_1(d-1)}{D_2(d-1)}$ which give the correct induction relation (see (28) and (30)). It is also easy to check that $D_4(d) = D_2(d-1)\sqrt{2}Sh_{d,d-1}$.

c. Let us check that $D_4(d)$ leads to the correct constant term in the denominator in the Th 5.1. We have

$$D_4(d) = \frac{\alpha^{(d-1)/2}}{\sqrt{2}C_1(d)}F_2(d-1)F_3(d-2)\cdots F_3(2)\sqrt{\frac{\alpha}{2}}E^d$$

and by induction, we obtain the coefficient of $\beta_{d-3}\beta_{d-4}\cdots\beta_1$ in $D_4(d)$ as

$$\begin{aligned} &\frac{\alpha^{(d-1)/2}\frac{\sqrt{\alpha}}{\sqrt{2}}E^d}{\sqrt{2}\cdot\frac{\sqrt{\alpha}}{\sqrt{2}}\frac{E^{d,d-1}}{Sh_{d,d-1}}\cdot\frac{\sqrt{\alpha}}{\sqrt{2}}\frac{E^{d-1,d-2}}{Sh_{d-1,d-2}}\cdots\frac{\sqrt{\alpha}}{\sqrt{2}}\frac{e^{\sqrt{\cdot}\rho_1}}{Sh_1}} \\ &= Sh_{d,d-1}\cdot\cdot Sh_1 2^{(d-2)/2}. \end{aligned}$$

Plugging into (23) leads to the correct final constant.

d. It remains to develop

$$\frac{D_1(d)}{D_2(d)} = \frac{G^{**}_{d-2}(\rho_d,\rho_d)}{F_2(d)} = C_3(d) + \frac{f(d-1)}{D_2(d)}$$

with $f(d-1) = C_2(d-1)D_4(d)$.
This gives

$$C_3(d) + C_2(d-1)\frac{F_2(d-1)}{G^{**}_{d-3}(\rho_{d-1},\rho_{d-1}) + \beta_{d-2}F_2(d-1)}$$

But

$$\begin{aligned}\frac{G^{**}_{d-2}(\rho_d,\rho_d)}{F_2(d)} &= \frac{1}{C_1(d)F_2(d)} - \frac{C_2(d)}{C_1(d)}\\ &= \frac{1-\frac{C_2(d)}{C_1(d)}G^{**}_{d-3}(\rho_{d-1},\rho_{d-1}) + \beta_{d-2}[G^{**}_{d-3}(\rho_{d-2},\rho_{d-2}) - \frac{C_2(d)}{C_1(d)}F_2(d-1)]}{G^{**}_{d-2}(\rho_{d-1},\rho_{d-1}) + \beta_{d-2}F_2(d-1)}\end{aligned}$$

Identification leads to

$$\begin{aligned}G^{**}_{d-3}(\rho_{d-2},\rho_{d-2}) - \frac{C_2(d)}{C_1(d)}F_2(d-1) &= C_3(d)F_2(d-1)\\ 1-\frac{C_2(d)}{C_1(d)}G^{**}_{d-3}(\rho_{d-1},\rho_{d-1}) &= C_3(d)G^{**}_{d-3}(\rho_{d-1},\rho_{d-1}) + C_2(d-1)F_2(d-1).\end{aligned}$$

But, by (26), $G^{**}_{d-3}(\rho_{d-2},\rho_{d-2}) = C_1(d-1)F_2(d-1)$.
So

$$C_3(d) = C_1(d-1) - \frac{C_2(d)}{C_1(d)} = \frac{\sqrt{\alpha}}{\sqrt{2}}\frac{Sh_{d,d-2}}{Sh_{d,d-1}Sh_{d-1,d-2}}$$

and

$$C_4(d)F_2(d-1) = 1 - C_1(d-1)G^{**}_{d-3}(\rho_{d-1},\rho_{d-1}) = C_2(d-1)F_2(d-1)$$

by (27), hence $C_4(d) = C_2(d-1)$ which is the correct value.

2. *Proceeding forwards by induction from $d-1$ to d:* This is just a series of substitutions. We omit the details.

□

Proof: *Proof of Lemma 5.1, Part B.*
We must now analyze $\bar{G}_d(a,b) - \bar{G}_{d-1}(a,b)$

$$= G_{d-1}(a,b) - \bar{G}_{d-1}(a,b) - \frac{G^{**}_{d-1}(a,\rho_d)G^{**}_{d-1}(b,\rho_d)}{G^{**}_{d-1}(\rho_d,\rho_d)}$$

This first part clearly leads to $\Theta(d-1)$.
The second part gives, by (22) and (23)

$$-\frac{\alpha^{d-1}Sh_aSh_b}{2F_1^2[C_1(d) + C_2(d)D_2(d)/F_1(d)]}$$

□

5.4.2 Using the random tree approach

Now we will use the results of section 2.3 in order to deduce Theorem 5.1. The proof consists of three parts: First we will evaluate the coefficient $[u_1^{k_1}\cdots u_d^{k_d}]g(z,u_1,\ldots,u_d)=F(z)$ in the right-hand side of (10). This can be done explicitly by solving recursion (9). Afterwards we will determine the main term of $[z^n]F(z)$ by means of Cauchy's formula and singularity analysis and in the last part we present the error estimates.

1. *Evaluating* $[u_1^{k_1}\cdots u_d^{k_d}]g(z,u_1,\ldots,u_d)$: By (9) we have

$$b_k(z,u)=z\frac{d_k(z)-ud_{k-1}(z)}{d_{k+1}(z)-ud_k(z)} \tag{31}$$

with

$$d_k(z)=\left(\frac{1+\sqrt{1-4z}}{2}\right)^k-\left(\frac{1-\sqrt{1-4z}}{2}\right)^k. \tag{32}$$

Thus we get

$$\begin{aligned} b_{r_1}(z,u_1b_{h_{12}}) &= z\frac{d_{r_1}-u_1b_{h_{12}}d_{r_1-1}}{d_{r_1+1}-u_1b_{h_{12}}d_{r_1}} \\ &= z\frac{d_{r_1}-u_1b_{h_{12}}d_{r_1-1}}{d_{r_1+1}}\sum_{i\ge 0}\left(\frac{d_{r_1}}{d_{r_1+1}}b_{h_{12}}\right)^i u_1^i, \end{aligned}$$

and consequently

$$[u_1^{k_1}]b_{r_1}(z,u_1b_{h_{12}}) = z\left(\frac{d_{r_1}}{d_{r_1+1}}-\frac{d_{r_1-1}}{d_{r_1}}\right)\left(\frac{d_{r_1}}{d_{r_1+1}}b_{h_{12}}\right)^{k_1}.$$

Applying (31) once more leads to

$$b_{h_{12}}(z,u_2b_{h_{23}}) = \frac{zd_{h_{12}}}{d_{h_{12}+1}}\left(1+\left(\frac{d_{h_{12}}}{d_{h_{12}+1}}-\frac{d_{h_{12}-1}}{d_{h_{12}}}\right)\frac{u_2b_{h_{23}}}{1-u_2b_{h_{23}}d_{h_{12}}/d_{h_{12}+1}}\right)$$

and thus we obtain by elementary calculations

$$\begin{aligned} [u_1^{k_1}u_2^{k_2}]b_{r_1}(z,u_1b_{h_{12}}(z,u_2b_{h_{23}})) &= z\left(1-\frac{d_{r_1-1}d_{r_1+1}}{d_{r_1}^2}\right)\left(\frac{d_{r_1}}{d_{r_1+1}}\right)^{k_1+1} \\ &\quad\times\left(\frac{zd_{h_{12}}}{d_{h_{12}+1}}\right)^{k_1}\left(\frac{bd_{h_{12}}}{d_{h_{12}+1}}\right)^{k_2}S(k_1,k_2,d_{h_{12}}) \end{aligned}$$

where

$$S(a,b,d_h)=\sum_{i=0}^{\min(a,b)-1}\binom{a}{i+1}\binom{b-1}{i}\left(1-\frac{d_{h-1}d_{h+1}}{d_h^2}\right)^{i+1}$$

Proceeding analogously yields

$$[u_3^{k_3}]b_{h_{23}}^{k_2}(z,u_3b_{h_{34}}) = \left(\frac{zd_{h_{23}}}{d_{h_{23}+1}}\right)^{k_2}\left(1+\left(\frac{d_{h_{23}}}{d_{h_{23}+1}}-\frac{d_{h_{23}-1}}{d_{h_{23}}}\right)\frac{u_2b_{h_{34}}}{1-u_2b_{h_{34}}d_{h_{23}}/d_{h_{23}+1}}\right)^{k_2}$$

$$
\begin{aligned}
&= \left(\frac{zd_{h_{23}}}{d_{h_{23}+1}}\right)^{k_2}\left(\frac{b_{h_{34}}d_{h_{23}}}{d_{h_{23}+1}}\right)^{k_3} S(k_2,k_3,d_{h_{23}})\\
&\vdots \quad \vdots \quad \vdots\\
[u_d^{k_d}]b_{h_{d-1,d}}^{k_{d-1}}(z,u_d a(z)) &= \left(\frac{zd_{h_{d-1,d}}}{d_{h_{d-1,d}+1}}\right)^{k_{d-1}}\left(\frac{a(z)d_{h_{d-1,d}}}{d_{h_{d-1,d}+1}}\right)^{k_d} S(k_{d-1},k_d,d_{h_{d-1,d}}).
\end{aligned}
$$

Therefore the desired coefficient satisfies $[u_1^{k_1}\cdots u_d^{k_d}]g(z,u_1,\ldots,u_d) = [z^n]F(z)$ with

$$
\begin{aligned}
F(z) &= z\left(1-\frac{d_{r_1-1}d_{r_1+1}}{d_{r_1}^2}\right)\left(\frac{d_{r_1}}{d_{r_1+1}}\right)^{k_1+1}\left(\frac{zd_{h_{12}}}{d_{h_{12}+1}}\right)^{k_1}\left(\frac{a(z)d_{h_{d-1,d}}}{d_{h_{d-1,d}+1}}\right)^{k_d} S(k_{d-1},k_d,d_{h_{d-1,d}})\\
&\times\prod_{l=2}^{d-1}\left[\left(\frac{d_{h_{l-1,l}}}{d_{h_{l-1,l}+1}}\right)^{k_l}\left(\frac{zd_{h_{l,l+1}}}{d_{h_{l,l+1}+1}}\right)^{k_l} S(k_{l-1},k_l,d_{h_{l-1,l}})\right] \qquad (33)
\end{aligned}
$$

2. *Applying Cauchy's formula and singularity analysis*: Now, in order to calculate the above coefficient we use Cauchy's integral formula choosing a truncated line normal to the real axis and complemented by a circular arc as integration path. To be precise, we integrate along $\Gamma=\gamma'\cup\Gamma'$ given by

$$
\begin{aligned}
\gamma' &= \left\{z : z=\frac{1}{4}\left(1-\frac{1+it}{n}\right) \text{ and } |t|\le\sqrt{2n+1}\right\}\\
\Gamma' &= \left\{z : |z|=\frac{1}{4} \text{ and } \arctan\frac{\sqrt{2n+1}}{n-1}\le|\arg z|\le\pi\right\}
\end{aligned}
$$

Let us analyze the factors (33). On γ' we substitute $z=\frac{1}{4}\left(1-\frac{\alpha}{n}\right)$. Analyzing the factors occurring in (33) asymptotically yields

$$
\begin{aligned}
1-\frac{d_{h_{l,l+1}-1}d_{h_{l,l+1}+1}}{d_{h_{l,l+1}}^2} &\sim \frac{1}{n}\frac{\alpha}{\sinh^2\left((\rho_{l+1}-\rho_l)\sqrt{\alpha}\right)} &(34)\\
\left(\frac{zd_{h_{l,l+1}}}{d_{h_{l,l+1}+1}}\right)^{k_l} &\sim 2^{-k_l}\exp\left(-\frac{y_l\sqrt{\alpha}\cosh\left((\rho_{l+1}-\rho_l)\sqrt{\alpha}\right)}{\sinh\left((\rho_{l+1}-\rho_l)\sqrt{\alpha}\right)}\right) &(35)\\
\left(\frac{a(z)d_{h_{l,l+1}}}{d_{h_{l,l+1}+1}}\right)^{k_l} &\sim \exp\left(-\frac{y_l\sqrt{\alpha}e^{(\rho_{l+1}-\rho_l)\sqrt{\alpha}}}{\sinh\left((\rho_{l+1}-\rho_l)\sqrt{\alpha}\right)}\right) &(36)
\end{aligned}
$$

and

$$
\left(\frac{d_{r_1}}{d_{r_1+1}}\right)^{k_1+1}\sim 2^{k_1+1}\exp\left(-\frac{y_1\sqrt{\alpha}\cosh\left(b_1\sqrt{\alpha}\right)}{\sinh\left(b_1\sqrt{\alpha}\right)}\right) \qquad (37)
$$

The remaining task is approximating $S(k_l,k_{l+1},d_{h_{l,l+1}})$. In order to do that we apply Stirling's formula and obtain for any $\varepsilon>0$

$$
\begin{aligned}
&\sum_{i\le n^{1/4-\varepsilon}}\binom{k_l}{i+1}\binom{k_{l+1}-1}{i}\left(1-\frac{d_{h_{l,l+1}-1}d_{h_{l,l+1}+1}}{d_{h_{l,l+1}}^2}\right)^i\\
&\quad\sim\sqrt{n}\sum_{i\le n^{1/4-\varepsilon}}\frac{y_l^{i+1}}{(i+1)!}\frac{y_{l+1}^i}{i!}\left(\frac{\alpha}{\sinh^2\left((\rho_{l+1}-\rho_l)\sqrt{\alpha}\right)}\right)^i \qquad (38)
\end{aligned}
$$

The remainder of the sum may be estimated by taking absolute values:

$$\left| \sum_{i \geq n^{1/4-\varepsilon}} \binom{k_l}{i+1} \binom{k_{l+1}-1}{i} \left(1 - \frac{d_{h_{l,l+1}-1} d_{h_{l,l+1}+1}}{d^2_{h_{l,l+1}}}\right)^i \right| \leq \sqrt{n} \sum_{i \geq n^{1/4-\varepsilon}} \frac{y_l^{i+1}}{i!} \frac{y_{l+1}^i}{i!} \left| \frac{\alpha}{\sinh^2\left((\rho_{l+1}-\rho_l)\sqrt{\alpha}\,\right)} \right|^i \tag{39}$$

The right-hand side of this equation is a sum of the form $\sum_{j \geq m} A^j/(j!)^2, A > 0$. By means of Stirling's formula we get the estimate

$$\left(\frac{A^{j/2}}{j!}\right)^2 \leq \exp\left(j(2 + \log A) - 2j \log j\right) = \exp\left(j\left(2 + \log \frac{A}{j^2}\right)\right) \tag{40}$$

By definition of γ' we have $\alpha = 1 + it$ and so elementary calculations yield $\Re\sqrt{\alpha} = \sqrt{(\sqrt{1+t^2}+1)/2}$. Now observe that $|\sinh^2 z|^2 = \sinh^2(\Re z) + \sin^2(\Im z)$ and that $\sinh^2 x \geq x^2$ for real x. Thus

$$A = \left| \frac{y_l y_{l+1} \alpha}{\sinh^2\left((\rho_{l+1}-\rho_l)\sqrt{\alpha}\,\right)} \right| \leq \left| 2 \frac{y_l y_{l+1} \sqrt{1+t^2}}{(\rho_{l+1}-\rho_l)^2 (\sqrt{1+t^2}+1)} \right| = \mathcal{O}\left(1\right)$$

As we are considering the case $j \geq n^{1/4-\varepsilon}$ in (40), the exponent satisfies $j(2 + \log(A/j^2)) \leq -C \log n$ for a suitable constant C which implies

$$\sum_{j \geq m} \frac{A^j}{(j!)^2} \leq \sum_{j \geq m} \left(\frac{1}{n^C}\right)^j = \mathcal{O}\left(e^{-Cm \log n}\right),$$

where $m = n^{1/4-\varepsilon}$. Therefore the sum (39) is exponentially small and thus negligible. Collecting the constants in (34)–(38) and applying

$$b_n = \frac{1}{n} \binom{2n-2}{n-1} \sim \frac{4^{n-1}}{\sqrt{\pi n^3}}$$

yields finally (20) as desired.

3. *Error estimates*: In order to complete the proof of the theorem we have to estimate the errors, i.e. the contributions of Γ' and $\gamma \setminus \gamma'$. On $\gamma \setminus \gamma'$ set again $\alpha = 1 + it$. As above we get $\Re\sqrt{\alpha} = \sqrt{(\sqrt{1+t^2}+1)/2} \geq \sqrt{t}/2$ and $|\alpha| \leq 2t$. Furthermore for sufficiently large real u the inequality $\sinh^2 u \geq \exp(2u)/5$ holds. Thus for any fixed $\eta > 0$ we have for sufficiently large t

$$\left| \frac{\alpha}{\sinh^2(\eta\sqrt{\alpha}\,)} \right| \leq \frac{|\alpha|}{\sinh^2(\eta \Re\sqrt{\alpha}\,)} \leq 10t \exp(-\eta\sqrt{t}\,)$$

and

$$\frac{e^{\eta\sqrt{\alpha}}}{\sinh(\eta\sqrt{\alpha}\,)} \to 1, \quad \text{as } t \to \infty.$$

Since the integrand in (20) consists only of terms of the above forms (the Bessel function can be expanded into such terms, see e.g. [24, p.355]) and thus it is exponentially small as $\alpha \to \pm i\infty$. Hence we may extend γ' to γ.

Now let us turn to the contribution of Γ'. Due to $n/b_n \sim \sqrt{\pi n}/4^{n-1}$ we have to show that

$$\left| \int_{\Gamma'} [u_1^{k_1} v^l] b_{r_1}(z, u_1 b_{h_{12}}) \frac{dz}{z^{n+1}} \right| = o\left(\frac{4^n}{\sqrt{n}} \right)$$

In the following let $c_1, c_2, \ldots$ denote suitably chosen positive constants. On Γ' we have $|\sqrt{1-4z}| \geq n^{-1/4}$ and $\arg \sqrt{1-4z} \leq \pi/4$. Thus

$$|1+\sqrt{1-4z}|^{\rho_1\sqrt{n}} \geq \exp(c_1 \rho_1 n^{1/4}) \quad \text{and} \quad |1-\sqrt{1-4z}|^{\rho_1\sqrt{n}} \leq \exp(-c_2 \rho_1 n^{1/4}).$$

Using these estimates we obtain by elementary calculations

$$\begin{aligned}
\left| \frac{z^{r_1-1}(1-4z)}{d_{r_1}^2} \right| &= \mathcal{O}\left(\exp(-c_3 \rho_1 n^{1/4}) \right) \\
\left| \frac{d_{h_{l,l+1}}}{d_{h_{l,l+1}+1}} \right|^{k_l} &\sim \left(\frac{2}{1+\sqrt{1-4z}} \right)^{k_l} = \mathcal{O}\left(2^{k_1} \exp(-c_4 y_l n^{1/4}) \right) \\
\left| \frac{z d_{h_{l,l+1}}}{d_{h_{l,l+1}+1}} \right|^{k_l} &= \mathcal{O}\left(2^{-k_1} \exp(-c_5 y_l n^{1/4}) \right).
\end{aligned}$$

This implies that the contribution of Γ' is negligibly small which completes the proof of Part A.

4. *Proof of Part B*: This part is immediate. One has just to compute the coefficients

$$\begin{aligned}
&[z^n u_1^0 \cdots u_d^0] g(z, u_1, \ldots, u_d) \\
&[z^n u_1^{k_1} u_2^0 \cdots u_d^0] g(z, u_1, \ldots, u_d) \\
&\vdots \\
&[z^n u_1^{k_1} \cdots u_{d-1}^{k_{d-1}} u_d^0] g(z, u_1, \ldots, u_d)
\end{aligned}$$

which is an easy exercise.

□

6 The moments

Let us for instance derive $E[\tau^+(\rho_1)]$ from Lemma 5.1 .

Computing $\left|\frac{\partial}{\partial\beta_1}\right|_{\beta_1=0}$, we obtain

$$\frac{1}{2\pi i} \int_S e^{\alpha} 2 e^{-2\rho_1 \sqrt{\cdot}} \sqrt{2\pi} d\alpha. \tag{41}$$

But it is well known that $\mathcal{L}_\alpha[f(u)] = e^{-\sqrt{\cdot}a}$ with $f(u) = \frac{e^{-a^2/2u}a}{\sqrt{2\pi}u^{3/2}}$.

Setting $a = 2\rho_1$, and $u = 1$, we obtain from (41)

$$E[\tau^+(\rho_1)] = e^{-2\rho_1^2} 4\rho_1$$

which is well known.

¿From (17), we see that μ_r, the order r moment, is given by

$$\mu_r = \frac{r!}{\sqrt{2\pi} i} \int_S \frac{e^\alpha 2\alpha 2^r \sinh(\rho_1\sqrt{\cdot})^r}{\sinh(\rho_1\sqrt{\cdot})\sqrt{\cdot}e^{\rho_1\sqrt{\cdot}}(\sqrt{\cdot}e^{\rho_1\sqrt{\cdot}})^r} d\alpha \tag{42}$$

By Takács [23, (32)], this is also given by

$$\begin{aligned} \mu_r = 2^{r+1} r! \rho_1^r \int_0^1 \cdot\cdot \int_0^1 (1 + v_1 + \cdot\cdot + v_{r+1}) \cdot \\ \cdot \exp[-2\rho_1^2(1 + v_1 + \cdot\cdot + v_{r-1})^2] dv_1 \cdot\cdot dv_{r-1} \end{aligned} \tag{43}$$

Identifications of (42) and (43) can be done as follows.
First, define

$$\mu_r(u) := 2^{r+1} r! \frac{\rho_1^r}{u^{3/2}} \int_0^1 \cdot\cdot \int_0^1 R \exp(-2\rho_1^2 R^2/u) dv_1 \cdot\cdot dv_{r-1}$$

where $R := 1 + v_1 \cdot\cdot + v_{r-1}$
$f_7(\alpha) := \mathcal{L}_\alpha[\mu_r(u)]$ is given by

$$\begin{aligned} f_7(\alpha) &= 2^{r+1} r! \rho_1^r \int_0^1 \cdot\cdot \int_0^1 \exp[-2\sqrt{\cdot}\rho_1 R] \frac{\sqrt{\pi}}{\sqrt{2}\rho_1} dv_1 \cdot\cdot dv_{r-1} \\ &= \frac{2^{r+1} r! \rho_1^r \sqrt{\pi}}{\sqrt{2}\rho_1} e^{-2\sqrt{\cdot}\rho_1} \left(\frac{1 - e^{-2\sqrt{\cdot}\rho_1}}{2\sqrt{\cdot}\rho_1} \right)^{r-1} \end{aligned}$$

Now, set $u = 1$, so

$$\mu_r = \mu_r(1) = \frac{1}{2\pi i} \int_S e^\alpha f_7(\alpha) d\alpha$$

Identification with (42) is immediate.
Using the same trick, we can now recover equ (47) of Takács [23]. Indeed this expression gives

$$\begin{aligned} & E[e^{-\beta_1 \tau^+(\rho_1)}] = \\ & 1 + 2\sum_{k=1}^\infty \frac{(2\beta_1\rho_1)^k}{(k-1)!} \int_k^\infty (1 - 4\rho_1^2 v^2)(v-k)^{k-1} \exp[-2\rho_1^2 v^2 - 2\rho_1(v-k)\beta_1] dv. \end{aligned}$$

Set $w = v - k$ and

$$\begin{aligned} f_8(u) := 2\sum_{k=1}^\infty \frac{(2\beta_1\rho_1)^k}{(k-1)!} \int_0^\infty [\frac{1}{u^{3/2}} - \frac{4\rho_1^2}{u^{5/2}}(w+k)^2] w^{k-1} \cdot \\ \cdot \exp[\frac{-2\rho_1^2}{u}(w+k)^2 - 2\rho_1 w \beta_1] dw. \end{aligned}$$

But we know that

$$\mathcal{L}_\alpha \left[\frac{e^{-a^2/2u}}{\sqrt{2\pi}} (\frac{1}{u^{3/2}} - \frac{a^2}{u^{5/2}}) \right] = -e^{-\sqrt{\cdot}a}\sqrt{\cdot}$$

So

$$f_9(\alpha) := \mathcal{L}_\alpha(f_8(u)) =$$
$$-2\sqrt{2\pi}\sqrt{\cdot}\sum_{k=1}^{\infty}\frac{(2\beta_1\rho_1)^k}{(k-1)!}\int_0^\infty w^{k-1}e^{-\sqrt{\cdot}2\rho_1(w+k)-2\rho_1 w\beta_1}dw$$
$$= -2\sqrt{2\pi}\sqrt{\cdot}\left[\frac{1}{1-\beta_1 e^{-2\rho_1\sqrt{\cdot}}/(\sqrt{\cdot}+\beta_1)}-1\right]$$

So we derive

$$E[e^{-\beta_1\tau^+(\rho_1)}]-1=\frac{1}{\sqrt{2\pi}i}\int_S e^\alpha\left[-2\sqrt{\cdot}\{\frac{\sqrt{\cdot}+\beta_1}{\sqrt{\cdot}+\beta_1(1-e^{-2\rho_1\sqrt{\cdot}})}-1\}\right]d\alpha$$

Identification with (16) is now immediate.

7 Conclusion

Using Kac's formula and a method of Chung on one hand and an approach via random trees on the other hand we have been able to derive a representation of the multi-dimensional Brownian excursion local time density which extends that of Hooghiemstra.

There remain two open problems: The first problem is to derive an explicit form of this density as has been done by Hooghiemstra [11] or Takács [23] for dimension 1. It seems to be very difficult to extend their methods to higher dimensions. Even direct proofs of the equivalence of all representations known for dimension 1 are (with one exception) unknown up to now.

Second, it would be desirable to have a general formula for the joint moments. In principle, the method presented in Sec. 6 works for calculating the joint moments. But for higher dimensions the expressions get quite involved such that a general formula seems not obtainable in this way. For instance, a straight forward calculation for $d=2$ gives

$$E[\tau^+(\rho_1)^k\tau^+(\rho_2)^l]=\frac{1}{i\sqrt{\pi}}\int\limits_S\left(\frac{2}{\alpha}\right)^{(k+l)/2}\frac{e^\alpha}{2}\sum_{i=0}^{\min(k,l)}\binom{l}{i}(-1)^i\frac{Sh_2^{l-i}Sh_{21}^{k+l+1}Sh_1^k(E^{2,1})^{k-i}}{(E^{2,1}Sh_2-Sh_1Sh_{21}^4)^{k+l+1-i}}\,d\alpha.$$

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Appendix

G_d satisfies many curious relations. Here are some of them. Proofs are by induction on m (checking for $m=1$ is immediate).

$$\frac{G_m^{**}(\rho_{k-1},\rho_l)}{G_m^{**}(\rho_k,\rho_l)} = E^{k,k-1}, k-1 \geq l > m \tag{1}$$

Proof: We derive

$$\frac{G_{m-1}^{**}(\rho_{k-1},\rho_l) + \beta_m[G_{m-1}^{**}(\rho_{k-1},\rho_l)G_{m-1}^{**}(\rho_m,\rho_m) - G_{m-1}^{**}(\rho_{k-1},\rho_m)G_{m-1}^{**}(\rho_l,\rho_m)]}{G_{m-1}^{**}(\rho_k,\rho_l) + \beta_m[G_{m-1}^{**}(\rho_k,\rho_l)G_{m-1}^{**}(\rho_m,\rho_m) - G_{m-1}^{**}(\rho_k,\rho_m)G_{m-1}^{**}(\rho_l,\rho_m)]}$$

Induction is trivial.

□

$$G_m^{**}(\rho_k,\rho_l)G_m^{**}(\rho_{k-1},\rho_v) - G_m^{**}(\rho_{k-1},\rho_l)G_m^{**}(\rho_k,\rho_v) = 0, k > \{l,v\} > m \tag{2}$$

Proof: Again, expanding the numerator induces a trivial induction.

□

$$\frac{G_m^{**}(\rho_k,\rho_k)G_m^{**}(\rho_{k-1},\rho_l) - G_m^{**}(\rho_k,\rho_{k-1})G_m^{**}(\rho_k,\rho_l)}{G_m^{**}(\rho_k,\rho_l)} = \frac{Sh_{k,k-1}\sqrt{2}}{\sqrt{\alpha}}, k > k-1 \geq l > m \tag{3}$$

Proof: Set $F_5 := G_{m-1}^{**}(\rho_k,\rho_k)G_{m-1}^{**}(\rho_{k-1},\rho_l) - G_{m-1}^{**}(\rho_k,\rho_{k-1})G_{m-1}^{**}(\rho_k,\rho_l)$

Expanding, and using (2) with $l = m, m' = m-1$, we derive after some algebra

$$\frac{F_5 + \beta_m F_7}{G_{m-1}^{**}(\rho_k,\rho_l) + \beta_m G_{m-1}^{**}(\rho_k,\rho_l)G_{m-1}^{**}(\rho_m,\rho_m) - \beta_m G_{m-1}^{**}(\rho_k,\rho_m)G_{m-1}^{**}(\rho_l,\rho_m)}$$

with $F_7 = F_5 G_{m-1}^{**}(\rho_m,\rho_m) - G_{m-1}^{**}(\rho_l,\rho_m)[G_{m-1}^{**}(\rho_k,\rho_k)G_{m-1}^{**}(\rho_{k-1},\rho_m) - G_{m-1}^{**}(\rho_k,\rho_{k-1})G_{m-1}^{**}(\rho_k,\rho_m)]$.

Again, induction is immediate.

□

$$G_m^{**}(a,\rho_k)G_m^{**}(\rho_s,\rho_t) - G_m^{**}(a,\rho_s)G_m^{**}(\rho_k,\rho_t) = 0, k > \{s,t\} > m \tag{4}$$

Proof: Again induction is obvious.

□