Generalized covariances of multi-dimensional Brownian Excursion local times

Guy Louchard * John W.Turner†

April 3, 2003

Abstract

Expressions for the generalized covariances of multi-dimensional Brownian excursion local times are derived from corresponding densities transforms. Typical applications are moments of the cost of structures such as $M/G/1$ queue, Random trees, Markov stack or priority queue in Knuth's model. Brownian excursion area and a result of Biane and Yor are also revisited.

1 Introduction

Throughout this paper, the standard Brownian motion (BM) will be denoted by $x(t)$. Three other classical BM are the reflecting BM: $x^+(t) \equiv |x(t)|$, the absorbing BM, killed at $0: x^-(t)$ and the absorbing, two-barriers $[0,\varepsilon]$ BM, killed at 0 and $\varepsilon : \tilde{x}(t,\varepsilon)$.

Fix $t > 0$ and denote the last zero of x before t and the first zero of x after t by

$$G(t) := \sup\{s : x \leq t; x(s) = 0\}$$

and

$$D(t) := \inf\{s : s \geq t; x(s) = 0\}.$$

The processes restricted to $[G(t), t]$ and $[G(t), D(t)]$ are called the meandering process ending at $t : Z(u) := x^+(G(t) + u), 0 \leq u \leq L^-(t) := t - G(t)$ and the excursion process straddling t : $Y(u) := x^+(G(t) + u), 0 \leq u \leq L(t) := D(t) - G(t)$, respectively. The standard scaled excursion (BE) is $X(u) := [Y(u)|L = 1]$; note that $Y(u) \stackrel{d}{\equiv} \sqrt{\ell}X(u/\ell)$ when $L = \ell$. The distributions of G and L are well known: see Chung [[3], Theorem 1].

The local time of $x(t)$ at a, denoted by

$$t^+(t,a) = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \int_0^t I_{[a,a+\epsilon]}(x(t))dt,$$

and the local time of the standard scaled excursion X at a, denoted by $\tau^+(a)$, have been studied by several authors (note that for an excursion of length ℓ we have: $\tau^+(\ell, a) \stackrel{d}{\equiv} \sqrt{\ell}\tau^+(a/\sqrt{\ell})$. See for instance Getoor and Sharpe [11], Knight [15], [16], Cohen and Hooghiemstra [4], Hooghiemstra [14], Drmota and Gittenberger [5], Louchard [19], Gittenberger and Louchard [13]. Intuitively, the local time at a is the total time spent by the excursion in the neighbourhood of a. Applications of the BE are numerous: we will mention a few of them, emphasizing the meaning of the local time. For instance, consider a $M/G/1$ queuing system. There the customers arrive according to Poisson process $(\pi_t, t \geq 0)$ with rate α^{-1} where $\alpha > 0$. Denote the arriving time of the n-th customer by t_n and the

*Université Libre de Bruxelles, Département d'Informatique, CP 212, Boulevard du Triomphe, B-1050 Bruxelles, Belgium, email: louchard@ulb.ac.be

†Université Libre de Bruxelles, Département de Physique, CP 231, Boulevard du Triomphe, B-1050 Bruxelles, Belgium, email:jwturner@ulb.ac.be

service time by s_n which is assumed to be independent of the arrival process π_t. Then the actual waiting time process is defined by $w_1 := 0, w_{n+1} := \max\{0, w_n + s_n - (t_{n+1} - t_n)\}$ and the virtual waiting time process by

$$v_t := \max\{0, w_{\pi_t} + s_{\pi_t} - (t - t_{\pi_t})\}, t \geq 0.$$

Furthermore, denote the length of the first busy period by ℓ. Then Cohen and Hooghiemstra [4] have shown that for arbitrary $\delta > 0$ the following limit theorem holds:

$$\left(\left(\frac{v_{su}}{\sqrt{2\alpha s}} \middle| s < \ell \leq s + \delta \right), 0 \leq u \leq 1 \right) \xrightarrow{d} X(u), \quad s \to \infty.$$

In this context the BE local time process appears as the weak limit of the (suitably normalized) number of downcrossings of the virtual waiting time process, i.e.

$$d(v) = \#\{t : 0 \leq t \leq \ell, v_t = v\},$$

($\#A$ denotes the cardinality of A) conditioned on the number of customers served during the first busy period (see [[4], Sec. 7]). Another BE application is the number of nodes at some level in a random tree. Consider a simply generated random tree (according to the notion of Meir and Moon [22] or, equivalently, the family tree of a Galton-Watson branching process conditioned on the total progeny. Then BE appears as the weak limit of the contour process of this tree, i.e. the process constructed of the distances of the nodes from the root when traversing the tree (for details see Gittenberger [12]). The local time corresponds here to the number of nodes at some level. The generation sizes of such branching processes converge weakly to BE local time. The external path length (EPL) of a random tree is given by the sum of distances from the root to the leafs.

Dynamical structures are also related the BE. The Stack structure of length $2n$ (see Flajolet [6] p. 126) is asymptotically equivalent to a BE (Louchard [20]). The priority queue in Knuth's model is combinatorially equivalent to a Markov Stack (see Louchard et al [21]). So the distribution of the size of this structure is asymptotically related, after suitable normalization, to the BE local time. The local time corresponds to the time spent by the structure at some level.

The cost G of structures such as $M/G/1$ queue busy period, random tree, Markov stack or priority queue in Knuth's model is asymptotically given, for any cost function $g(\cdot)$, by $G = \int_0^1 g[X(u)]du$. For stacks and priority queue, the cost is related to the size. For the $M/G/1$ queue, the cost is related to the waiting time. For EPL, the cost is related to the distance to the leafs.

Moments of G are immediately related to the local time: we have

$$\mathbb{E}[G^d] = d! \int_0^\infty dx_1 \int_{x_1}^\infty dx_2 \ldots \int_{x_{d-1}}^\infty dx_d \quad g(x_1)g(x_2)\ldots g(x_d) \cdot K(x_1, x_2, \ldots, x_d),$$

with $K(x_1, x_2, \ldots, x_d) := \mathbb{E}[\tau^+(x_1)\tau^+(x_2)\ldots\tau^+(x_d)], \quad x_1 \leq x_2 \ldots \leq x_d$, denoting the generalized covariances. In this paper we obtain explicit expressions for $K(x_1, x_2, \ldots, x_d)$. As an exercice, we revisit two well-known processes: the area of the BE ($g(x) = x$) and a result of Biane and Yor [2] related to $g(x) = 1/x$.

The paper is organized as follows. Sec. 2 gives the basic formula's we need in the sequel, Sec. 3 provides an efficient algorithm for the generalized covariances computation. In Sec. 4, we consider two typical applications: the Brownian excursion area and the Biane and Yor formula. Sec. 5 concludes the paper. An appendix shows some useful ζ-related identities.

A preliminary version of this paper was presented at LATIN 2000.

2 Basic formulae

In this section, we start from known results to derive expressions for the first generalized covariances $K(x_1, x_2, \ldots, x_d), d = 1 \ldots 4$.

In [13] we obtained the following results depending on some Laplace transforms.

$$\int_0^\infty e^{-\alpha t}\mathbb{E}\left[e^{-\sum_1^d \beta_i\sqrt{t}\tau^+(x_i/\sqrt{t})}, t > m_{x_d}\right]\frac{dt}{\sqrt{2\pi t^3}} = \Theta(d), \quad x_d > x_{d-1}\ldots > x_1$$

hence

$$\begin{aligned}\int_0^\infty e^{-\alpha t}\mathbb{E}\left[\tau^+(x_1/\sqrt{t})\cdots\tau^+(x_d/\sqrt{t})\right]t^{(d-3)/2}\frac{dt}{\sqrt{2\pi}} &= \left|\frac{\partial\Theta(d)}{\partial\beta_1\cdots\partial\beta_d}\right|_{\beta_1\cdots\beta_d=0} \\ &= \bar{K}(\alpha, x_1\ldots x_d), \quad \text{say}\ , \quad x_d > x_{d-1}\ldots > x_1,\end{aligned}$$

and

$$\begin{aligned}\mathbb{E}[e^{-\sum_1^d\beta_i\tau^+(x_i)}, 1 > m_{x_d}] &= \frac{1}{\sqrt{2\pi i}}\int_S e^\alpha\Theta(d)d\alpha, \quad x_d > x_{d-1}\ldots > x_1,\\ K(x_1,x_2,\ldots,x_d) &= \frac{1}{\sqrt{2\pi i}}\int_S e^\alpha\bar{K}(\alpha, x_1\ldots x_d)d\alpha, \quad x_d > x_{d-1}\ldots > x_1,\end{aligned}$$

where $S := [a - i\infty, a + i\infty], a > 0, m_x := \inf\{s : X(s) = x\}$,

$$\begin{aligned}\Theta(1) &= \frac{2\alpha}{Sh(\beta_1\sqrt{\cdot})\left[2\beta_1 Sh(\beta_1\sqrt{\cdot}) + \sqrt{\cdot}e^{\beta_1\sqrt{\cdot}}\right]},\\ \Theta(d) &= \frac{\alpha^d}{2[F_1(d)]^2[\beta_d + C_1(d) + C_2(d)D_2(d)/F_1(d)]}, \quad d \geq 2, \end{aligned} \qquad (1)$$

with some functions depending only on α and x.:

$$\begin{aligned}C_1(d) &= \sqrt{\frac{\alpha}{2}}E(d,d-1)/Sh(d,d-1), \quad d \geq 2,\\ C_2(d) &= -\frac{\alpha}{2Sh^2(d,d-1)}, \quad d \geq 2,\\ C_3(d) &= \sqrt{\frac{\alpha}{2}}\frac{Sh(d,d-2)}{Sh(d,d-1)Sh(d-1,d-2)}, \quad d \geq 3,\\ C_4(d) &= C_2(d-1), \quad d \geq 3,\\ C_5(d) &= \sqrt{2}Sh(d,d-1), \quad d \geq 3,\end{aligned}$$

and some functions depending also on β.:

$$\begin{aligned}F_1(d) &= \beta_{d-1}D_2(d) + D_1(d), \quad d \geq 2,\\ D_2(d) &= \beta_{d-2}D_4(d) + D_3(d), \quad d \geq 3,\\ \frac{D_1(d)}{D_2(d)} &= C_3(d) + C_4(d)D_4(d)/D_2(d), \quad d \geq 3,\\ D_3(d) &= C_5(d)D_1(d-1), \quad d \geq 3,\\ D_4(d) &= C_5(d)D_2(d-1), \quad d \geq 3,\end{aligned}$$

and

$$\begin{aligned}E(\ell,m) &:= e^{\sqrt{\cdot}[x_\ell - x_m]},\\ Sh(\ell,m) &:= \sinh[\sqrt{\cdot}(x_\ell - x_m)], Sh(\ell) := \sinh(\sqrt{\cdot}x_\ell),\\ \sqrt{\cdot} &:= \sqrt{2\alpha}.\end{aligned}$$

Initialisations are given by

$$\begin{aligned}D_1(2) &= \sqrt{\alpha}Sh(2)/\sqrt{2},\\ D_2(2) &= Sh(1)Sh(2,1).\end{aligned}$$

From (1), it is possible (with MAPLE) to derive explicit expressions for successive derivaties of $\Theta(d)$. This leads to Laplace transforms of $K(x_1, x_2, \ldots, x_d)$. For instance, with $E(i) := e^{\sqrt{\cdot} x_i}$,
$x_1 \leq x_2 \ldots \leq x_d$,

$$\begin{aligned}
\bar{K}(\alpha, x_1) &= \left|\frac{\partial\Theta}{\partial\beta_1}\right|_{\beta_1=0} = 2E(1)^{-2}, \\
\bar{K}(\alpha, x_1, x_2) &= \left|\frac{\partial^2\Theta}{\partial\beta_2\partial\beta_1}\right|_{\beta_1,\beta_2=0} = \frac{-2\sqrt{2}}{\sqrt{\alpha}}E(2)^{-2}(E(1)^{-2}-1), \qquad (2)\\
\bar{K}(\alpha, x_1, x_2, x_3) &= \left|\frac{\partial^3\Theta}{\partial\beta_3\partial\beta_2\partial\beta_1}\right|_{\beta_1,\beta_2,\beta_3=0} = \\
& 2\frac{[3E(1)^{-2}E(2)^{-2} - E(2)^{-2} - 2E(1)^{-2}](E(1)^{-2}-1)E(3)^{-2}}{E(1)^{-2}\alpha} \qquad (3)
\end{aligned}$$

$$\begin{aligned}
\bar{K}(\alpha, x_1, x_2, x_3, x_4) = &\left|\frac{\partial\Theta}{\partial\beta_4\partial\beta_3\partial\beta_2\partial\beta_1}\right|_{\beta_1,\beta_2,\beta_3,\beta_4=0} = \\
&-2\sqrt{2}E(4)^{-2}(E(1)^{-2}-1)\cdot \\
&\cdot[6E(1)^{-2}E(2)^{-4}E(3)^{-2} - 6E(1)^{-2}E(3)^{-2}E(2)^{-2} + 2E(1)^{-2}E(2)^{-2} + E(2)^{-4} \\
&+E(1)^{-2}E(3)^{-2} - 3E(1)^{-2}E(2)^{-4} + 2E(3)^{-2}E(2)^{-2} - 3E(2)^{-4}E(3)^{-2}]/ \\
&[E(2)^{-2}E(1)^{-2}\alpha^{3/2}].
\end{aligned}$$

High order derivatives become difficult to compute, even with MAPLE. So another technique is obviously needed.

3 An efficient algorithm for generalized covariances computation

In this section, we first derive a recurrence equation for some functions arising in the generalized covariances. This leads to some differential equations for related exponential generating functions. A simple matrix representation is finally obtained and it remains to invert the Laplace transforms.

3.1 A recurrence equation

Let us first differentiate Θ w.r.t. β_d (each time, after differentiation w.r.t. β_i, we set $\beta_i = 0$). This gives

$$\frac{-\alpha^d}{2[C_1F_1 + C_2D_2]^2}.$$

We should write $C_1(d)$, etc. but we drop the $d-$ dependency to ease notations. Differentiating now w.r.t. β_{d-1}, this leads to

$$\frac{\alpha^d C_1 D_2}{[C_1D_1 + C_2D_2]^3} = \frac{\alpha^d C_1[\beta_{d-2}D_4 + D_3]}{[C_7\beta_{d-2}D_4 + C_7D_3 + C_1C_4D_4]^3},$$

with $C_7(d) := C_1(d)C_3(d) + C_2(d) = C_1(d)C_1(d-1)$ after detailed computation.

It is clear that the next differentiations will lead to some pattern. Indeed, set

$$H(d,i) := \frac{\partial^{d-2}}{\partial\beta_{d-2}\ldots\partial\beta_1}\left.\frac{D_2(d)^i}{[C_1(d)D_1(d) + C_2(d)D_2(d)]^{i+2}}\right|_{\beta_{d-2}\cdots\beta_1=0}. \qquad (4)$$

Obviously

$$\left|\frac{\partial^d\Theta}{\partial\beta_d\ldots\partial\beta_1}\right|_{\beta_d\cdots\beta_1=0} = C_1(d)\alpha^d(-1)^dH(d,1), \quad d \geq 3. \qquad (5)$$

Expanding (4) we derive (omitting the details)

$$\begin{aligned}H(d,i) &= \frac{1}{C_1(d)^{i+2}C_1(d-1)^{i-1}C_5(d)^2}\cdot\\ &\cdot\sum_{j=0}^{i-1}\binom{i-1}{j}(-1)^{i-1-j}C_2(d-1)^{i-1-j}[-2H(d-1,i-j)\\ &+(i+2)C_2(d-1)H(d-1,i-j+1)].\end{aligned} \tag{6}$$

Equ.(6) is still too complicated. So we set first $H_1(d,i) := H(d,i)C_2(d)^{i-1}$. This leads to

$$\begin{aligned}H_1(d,i) &= \frac{C_2(d)^{i-1}}{C_1(d)^{i+2}C_1(d-1)^{i-1}C_5(d)^2}\cdot\\ &\cdot\sum_{j=0}^{i-1}\binom{i-1}{j}(-1)^{i-1-j}[-2H_1(d-1,i-j)+(i+2)H_1(d-1,i-j+1)].\end{aligned}$$

But we remark that

$$\frac{C_2(d)}{C_1(d)C_1(d-1)} = -\frac{C_6(d-1)}{C_6(d)},$$

with

$$C_6(d) := E(d)^2 - E(d-1)^2. \tag{7}$$

Then, we set $H_2(d,i) := H_1(d,i)C_6(d)^{i-1}$ and we obtain

$$\begin{aligned}H_2(d,i) &= \frac{1}{C_1(d)^3C_5(d)^2}\cdot\\ &\cdot\sum_{j=0}^{i-1}\binom{i-1}{j}(-1)^{j}C_6(d-1)^{j}[-2H_2(d-1,i-j)\\ &+\frac{(i+2)}{C_6(d-1)}H_2(d-1,i-j+1)].\end{aligned} \tag{8}$$

3.2 Some generating function

Equ.(8) is a perfect candidate for an exponential generating function (see Flajolet and Sedgewick [10]). We set

$$\varphi_2(d,v) := \sum_{1}^{\infty}\frac{H_2(d,i)v^{i-1}}{(i-1)!}.$$

Equ.(8) leads to

$$\begin{aligned}&\varphi_2(d,v) =\\ &\quad\frac{1}{C_1(d)^3C_5(d)^2}\cdot\left\{\left[-2\varphi_2(d-1,v)+\frac{1}{C_6(d-1)}\frac{\partial^2}{\partial v^2}[\varphi_2(d-1,v)\cdot v]\right]e^{-vC_6(d-1)}\right.\\ &\quad\left.+\frac{1}{C_6(d-1)}\partial_v\varphi_2(d-1,v)\cdot\partial_v[e^{-vC_6(d-1)}\cdot v]\right\}.\end{aligned}$$

With (7), we are led to set

$$\varphi_3(d,v) := \varphi_2(d,v)e^{vE(d-1)^2} \text{ and } H(d,1) = \varphi_3(d,0).$$

Before establishing the corresponding equation for φ_3, it is now time to find the effect of all our transforms on φ_2. Indeed, $H(2,i) = \gamma\delta_3^{i-1}$ (see (4)) with

$$\begin{aligned}
\delta_1 &= D_2(2), \delta_2 = C_1(2)D_1(2) + C_2(2)D_2(2), \\
\gamma &= \frac{\delta_1}{\delta_2^3} = \frac{-2E(2)^{-2}(E(1)^{-2} - E(2)^{-2})(E(1)^{-2} - 1)}{E(1)^{-2}\alpha^3}, \\
\delta_3 &= \frac{\delta_1}{\delta_2}.
\end{aligned}$$

So

$$\begin{aligned}
H_1(2,i) &= \gamma\delta_4^{i-1}, \text{ with } \delta_4 = \delta_3 C_2(2), \\
H_2(2,i) &= \gamma\delta_5^{i-1}, \text{ with } \delta_5 = \delta_4 C_6(2), \\
\varphi_2(2,v) &= \gamma e^{v\delta_5}, \varphi_3(2,v) = \gamma e^{v\delta_6}, \text{with } \delta_6 = \delta_5 + E(1)^2 = 1,
\end{aligned}$$

after all computations.

We see that it is convenient to finally set

$$\varphi_3(d,v) := \varphi_4(d,v)e^v, \varphi_4(2,v) = \gamma, H(d,1) = \varphi_4(d,0).$$

The differential equation for φ_4 is computed as follows (we omit the details)

$$\begin{aligned}
\varphi_4(d,v) = &\frac{2(E(d-1)^{-2} - E(d)^{-2})E(d)^{-2}}{E(d-1)^{-4}(E(d-2)^{-2} - E(d-1)^{-2})\sqrt{\cdot}^3} \cdot \\
&\left[\mu_1 v \frac{\partial^2}{\partial v^2}\varphi_4(d-1,v) + (\mu_2 + \mu_3 v)\frac{\partial}{\partial v}\varphi_4(d-1,v) + (\mu_4 + \mu_5 v)\varphi_4(d-1,v)\right],
\end{aligned} \tag{9}$$

with

$$\begin{aligned}
\mu_1 &:= E(d-2)^{-2}E(d-1)^{-2}, \\
\mu_2 &:= 3E(d-2)^{-2}E(d-1)^{-2}, \\
\mu_3 &:= 2E(d-2)^{-2}E(d-1)^{-2} - E(d-2)^{-2} - E(d-1)^{-2}, \\
\mu_4 &:= -2E(d-2)^{-2} - E(d-1)^{-2} + 3E(d-2)^{-2}E(d-1)^{-2}, \\
\mu_5 &:= E(d-2)^{-2}E(d-1)^{-2} - E(d-1)^{-2} - E(d-2)^{-2} + 1.
\end{aligned}$$

3.3 A matrix representation

It is now clear that $\varphi_4(d,v)$ is made of 2 parts: the first one is given by the product of γ with all coefficients in front of (9).

The other part is given, for each d, by a polynomial in v, the coefficients of which are given by the following algorithm.

Start with $vec_2[0] = 1, vec_2[i] = 0, i \geq 1$.

Construct a tri-diagonal band matrix A_d as follows: if we apply the differential operator of (9),i.e. $\left[\mu_1 v \frac{\partial^2}{\partial v^2}\varphi_4(d-1,v) + (\mu_2 + \mu_3 v)\frac{\partial}{\partial v}\varphi_4(d-1,v) + (\mu_4 + \mu_5 v)\varphi_4(d-1,v)\right]$ to a polynomial $\sum_0 a_i v^i$, we see that the new polynomial $\sum_0 \bar{a}_i v^i$ is given by

$$\begin{aligned}
\bar{a}_0 &= A_d[0,1]a_1 + A_d[0,0]a_0, \\
\bar{a}_i &= A_d[i,i+1]a_{i+1} + A_d[i,i]a_i + A_d[i,i-1]a_{i-1}, i \geq 1,
\end{aligned}$$

with

$$\begin{aligned}
A_d[i,i+1] &:= [i(i+1) + 3(i+1)]E(d-1)^{-2}E(d-2)^{-2}, \\
A_d[i,i] &:= [(2i+3)E(d-1)^{-2}E(d-2)^{-2} - (i+2)E(d-2)^{-2} - (i+1)E(d-1)^{-2}], \\
A_d[i,i-1] &:= [E(d-1)^{-2}E(d-2)^{-2} - E(d-1)^{-2} - E(d-2)^{-2} + 1].
\end{aligned} \tag{10}$$

All other elements of A_d are set to 0.

Successive applications of A_ℓ to vec_2 give the coefficients of the polynomial part of $\varphi_4(d,v)$:

$$vec_d := \prod_{\ell=3}^{d} A_\ell \; vec_2. \tag{11}$$

Now, by (5), $|\frac{\partial^d \Theta}{\partial\beta_d \dots \partial\beta_1}|_{\beta_d \dots \beta_1 = 0} = C_1(d)\alpha^d(-1)^d \varphi_4(d,0) = C_8(d)\; vec_d[0]$, where $C_8(d)$, after all simplifications, is given, with (9) by

$$C_8(d) = \frac{-4(-1)^d}{\sqrt{\cdot}^{d-1}} \frac{E(d)^{-2}(E(1)^{-2}-1)}{E(1)^{-2}\dots E(d-2)^{-2}}, \quad d \geq 3. \tag{12}$$

Let us summarize our results in the following theorem

Theorem 3.1 $\bar{K}(\alpha, x_1, x_2, \dots, x_d) = |\frac{\partial^d \Theta}{\partial\beta_d \dots \partial\beta_1}|_{\beta_d \dots \beta_1 = 0} = C_8(d)\; vec_d[0]$ *where* $C_8(d)$ *is given by (12),* $vec_d = \prod_{\ell=3}^{d} A_\ell \; vec_\ell$*, with* $vec_2[0] = 1,\; vec_2[i] = 0,\; i \geq 1$ *and the band matrix* A_ℓ *is given by (10).*

The computation of our covariances is now trivial.

3.4 Inverting the Laplace transforms

It is well known that $\mathcal{L}_\alpha[f(u)] = e^{-\sqrt{\cdot}a}$, with $f(u) = \frac{e^{-a^2/2u}a}{\sqrt{2\pi}u^{3/2}}$. Also $\mathcal{L}_\alpha[g(u)] = \frac{e^{-\sqrt{\cdot}a}\sqrt{2}}{2\sqrt{\alpha}}$, with $g(u) = \frac{e^{-a^2/2u}}{\sqrt{2\pi u}}$. Hence, from (2),

$$\mathbb{E}[\tau^+(x_1)\tau^+(x_2)] = -4[e^{-2(x_1+x_2)^2} - e^{-2x_2^2}], \quad x_2 \geq x_1. \tag{13}$$

We recover immediately Cohen and Hooghiemstra [4], (6.15)

Similarly, from (3),

$$\begin{aligned}
&\mathbb{E}[\tau^+(x_1)\tau^+(x_2)\tau^+(x_3)] \\
&= 4\int_0^1 \Big\{3\left[e^{-2[x_1+x_2+x_3]^2/t}(x_1+x_2+x_3) - e^{-2[x_2+x_3]^2/t}(x_2+x_3)\right] \\
&- \left[e^{-2[x_2+x_3]^2/t}(x_2+x_3) - e^{-2[x_3+x_2-x_1]^2}(x_2+x_3-x_1)\right] \\
&-2\left[e^{-2[x_1+x_3]^2/t}(x_1+x_3) - e^{-2[x_3]^2/t}(x_3)\right]\Big\}\frac{dt}{t^{3/2}}.
\end{aligned} \tag{14}$$

Next covariances lead to similar expressions, with multiple integrals on t.

4 Some applications

In this section, we apply the generalized covariances to two classical problems: the Brownian Excursion Area, with cost function $g(x) = x$ and a result of Biane and Yor related to the cost function $g(x) = 1/x$. We can proceed either from the Laplace transforms or from the explicit covariances.

4.1 Brownian excursion area

In [19], [18] we proved that $W_d := \mathbb{E}[\int_0^1 X(u)du]^d$ satisfies the following recurrence equation. Let $\gamma_k := (36\sqrt{2})^k W_k \Gamma(\frac{2k-1}{2})/2\sqrt{\pi}$.

Then

$$\gamma_n = \frac{12n}{6n-1}\varphi_n - \sum_{k=1}^{n-1}\varphi_k \binom{n}{k}\gamma_{n-k}, \tag{15}$$

with $\varphi_k := \Gamma(3k+\frac{1}{2})/\Gamma(k+\frac{1}{2})$.

The corresponding distribution, also called the Airy distribution has been the object of recent renewed interest (see Spencer [23], Flajolet, Poblete and Viola [9], where many examples are given, Flajolet and Louchard [7]).

Equ.(15) leads to $W_1 = \sqrt{2\pi}/4, W_2 = 5/12, W_3 = \sqrt{2\pi}15/128...$ From (2) we compute the Laplace transform

$$2\int_0^\infty x_1 dx_1 \int_{x_1}^\infty x_2 dx_2 \bar{K}(\alpha, x_1, x_2) = 2\int_0^\infty x_1 dx_1 \int_{x_1}^\infty x_2 dx_2 \frac{-2\sqrt{2}}{\sqrt{\alpha}} E(2)^{-2}[E(1)^{-2} - 1] = \frac{5\sqrt{2}}{32\alpha^{5/2}},$$

and

$$2\int_0^\infty x_1 dx_1 \int_{x_1}^\infty x_2 dx_2 K(x_1, x_2) = \frac{1}{\sqrt{2\pi i}} \int_S e^\alpha 2 \int_0^\infty x_1 dx_1 \int_{x_1}^\infty x_2 dx_2 \bar{K}(\alpha, x_1, x_2) d\alpha. \tag{16}$$

From now on, we do not need (or use) the $t-$ dependency anymore. Inverting (16), this leads to $5/12$ as expected.

Similarly with $G(\alpha)$ given by (3), $3! \int_0^\infty x_1 dx_1 \int_{x_1}^\infty x_2 dx_2 \int_{x_2}^\infty x_3 dx_3 G(\alpha) = \frac{6.15}{128\alpha^4}$. Inverting, this leads to $\sqrt{2\pi}15/128$ as expected.

An interesting open problem is to derive the recurrence (15) from the matrix representation given by Thm. 1.

4.2 A formula of Biane and Yor

In [2], Biane and Yor proved that

$$Y := \int_0^1 \frac{dt}{X(t)} \stackrel{\mathcal{D}}{\equiv} 2\xi, \text{ where } \xi := \sup_{[0,1]} X(t).$$

As $\mathbb{E}[\tau^+(x_1)] = e^{-2x_1^2} 4x_1$, we immediately check that the cost $g(x) = 1/x^{2-\varepsilon}, \varepsilon > 0$ is the ultimate form of costs of type $x^{-\gamma}$. $\gamma \geq 2$ leads to a singularity at the origin.

It is well known that the distribution function of $\sup_{[0,1]} X(t)$ is given by $\Pr[\sup X(t) \leq \xi] = F_2(2\xi^2)$, where $F_2(x) = 1 + 2\sum_{n=1}^\infty (1 - 2n^2 x) e^{-n^2 x}$ and

$$\mathcal{L}(f_2) = \frac{(\pi\sqrt{\alpha})^2}{sh^2(\pi\sqrt{\alpha})}, \tag{17}$$

(see for instance Chung [3]).

From (17), the moment of even order of ξ are given by

$$\mathbb{E}[\xi^r] = r(r-1)\Gamma(\frac{r}{2})\zeta(r)/2^{r/2}. \tag{18}$$

It is proved in Flajolet and Odlyzko [8] that the same formula holds for all moments. For instance,

$$\begin{aligned} \mathbb{E}(\xi) &= \sqrt{\frac{\pi}{2}}, \mathbb{E}(\xi^2) = \frac{\pi^2}{6}, \mathbb{E}(\xi^3) = \frac{3\sqrt{2\pi}}{4}\zeta(3), \\ \mathbb{E}(\xi^4) &= \pi^4/30. \end{aligned}$$

To derive these expressions from our generalized covariances, we may proceed either from the Laplace transforms or from the explicit expressions. Both techniques lead of course to the same results but via different identities.

4.2.1 Using the explicit expressions

Dimension 2

Equ.(13) leads to

$$\mathbb{E}(Y^2) = -8\int_0^\infty \frac{dx_1}{x_1}\int_{x_1}^\infty \frac{dx_2}{x_2}[e^{-(x_1+x_2)^2} - e^{-x_2^2}].$$

Set

$$I(\gamma) := \int_0^\infty \frac{dx_1}{x_1}\int_{x_1}^\infty \frac{dx_2}{x_2}[e^{-(x_1+x_2)^2} - e^{-x_2^2}]e^{-\gamma x_1 x_2},$$

so

$$\frac{-dI(\gamma)}{d\gamma} = \int_0^\infty dx_1\int_{x_1}^\infty dx_2[e^{-(x_1+x_2)^2} - e^{-x_2^2}]e^{-\gamma x_1 x_2}.$$

Setting $x_2 = x_1 u, s = x_1^2$, we obtain $I(0) = \int_0^\infty d\gamma[\frac{-dI(\gamma)}{d\gamma}] = -\int_1^\infty \frac{du}{u}\ln(1+\frac{1}{u}) = -\frac{1}{2}\zeta(2) = -\pi^2/12$ by (A.1).

Finally $\mathbb{E}(Y^2) = -8(\frac{-\pi^2}{12}) = 4\frac{\pi^2}{6} = 4\mathbb{E}(\xi^2)$ as it should be.

Dimension 3

Equ.(14) leads to

$$\begin{aligned}\mathbb{E}(Y^3) = 3!2\sqrt{2}\int_0^1 \frac{dt}{t^{3/2}}\int_0^\infty \frac{dx_1}{x_1}\int_{x_1}^\infty \frac{dx_2}{x_2}\int_{x_2}^\infty \frac{dx_3}{x_3}\\ \left\{-2[e^{-[x_1+x_3]^2/t}(x_1+x_3) - e^{-x_3^2/t}x_3] + \text{ 2 similar terms }\right\}.\end{aligned} \tag{19}$$

Set $t = 1/s^2, x_3 = vx_2, x_2 = ux_1$, we derive for the integrals on the bracket

$$2\int_0^\infty \frac{dx_1}{x_1}\int_1^\infty \frac{du}{u}\int_1^\infty \frac{dv}{v}\int_1^\infty ds[x_1(1+uv)e^{-s^2x_1^2(1+uv)^2} - x_1 uve^{-s^2x_1^2u^2v^2}].$$

The inner integral can be replaced by

$$-\int_0^1 ds[\quad] = \int_{x_1(1+uv)}^{x_1uv} d\sigma e^{-\sigma^2} = x_1\int_{1+uv}^{uv} dwe^{-x_1^2w^2}.$$

Next we obtain

$$\begin{aligned}&3!2\sqrt{2}\left\{-2\left[2\int_1^\infty \frac{du}{u}\int_1^\infty \frac{dv}{v}\int_{1+uv}^{uv} dw\frac{\sqrt{\pi}}{2w}\right] + \ldots\right\}\\ &= 3!2\sqrt{2}\left\{-2\left[-\sqrt{\pi}\int_1^\infty\int_1^\infty \frac{du}{u}\frac{dv}{v}\ln\left(1+\frac{1}{uv}\right)\right] + \ldots\right\}.\end{aligned}$$

The two other terms in (19) lead to similar integrals and we finally derive

$$\begin{aligned}&-3!2\sqrt{2}\sqrt{\pi}\int_1^\infty \frac{du}{u}\int_1^\infty \frac{dv}{v}\left\{-2\ln\left(1+\frac{1}{uv}\right) - \ln\left(1+\frac{1}{uv+v-1}\right) + 3\ln\left(1+\frac{1}{uv+v}\right)\right\}\\ &= 3!2\sqrt{2}\sqrt{\pi}\frac{\zeta(3)}{2} \quad \text{by (A.2)}\\ &= 8\mathbb{E}(\xi^3),\end{aligned}$$

as it should be.

Some open problems are: to understand why only terms extracted from Thm. 1 lead to ζ functions, and also to explain the appendix identities and why they finally lead to the simple expression (18).

Other dimensions

For the even moments, we are led to integrals of the form

$$\int_0^\infty \frac{dx_1}{x_1}\int_{x_1}^\infty \frac{dx_2}{x_2}\dots\int_0^1 dt_0\int_0^{t_0} dt_1\dots\int_0^{t_i}\frac{dt}{\sqrt{t}}e^{-(x_k+x_j+\dots)^2/t}.$$

Set $x_1 = x_1'\sqrt{t}\dots x_d = x_d'\sqrt{t}$.

The dependence on t disappears in the x-integral (the t-integrals are easily computed). We are back to the Dim - 2 type of integrals.

For odd moments, we are led to integrals of the form

$$\int_0^\infty \frac{dx_1}{x_1}\int_{x_1}^\infty \frac{dx_2}{x_2}\dots\int_0^1 dt_0\int_0^{t_0} dt_1\dots\int_0^{t_i}\frac{dt}{t^{3/2}}e^{-(x_k+x_j+\dots)^2/t}(x_k+x_j+\dots).$$

Set $x_1 = x_1'\sqrt{t_i}\dots x_d = x_d'\sqrt{t_i}$ and $t=\tau t_i$.

This gives

$$\int_0^\infty \frac{dx_1'}{x_1'}\int_{x_1'}^\infty \frac{dx_2'}{x_2'}\dots\int_0^1 dt_0\int_0^{t_0} dt_1\dots\int_0^{1} d\tau\frac{e^{-(x_k'+x_j'+\dots)^2/\tau}(x_k'+x_j'+\dots)}{\tau^{3/2}},$$

and we are back to the Dim-3 type of integrals.

4.2.2 Using the Laplace transforms

Dimension 2

Equ.(2) leads to

$$\frac{-4\sqrt{2}}{\sqrt{\alpha}}\int_0^\infty \frac{dx_1}{x_1}\int_{x_1}^\infty \frac{dx_2}{x_2}e^{-2\sqrt{\cdot}x_2}[e^{-2\sqrt{\cdot}x_1}-1].$$

Let us analyze a more general form

$$I(\delta,\beta) := \int_0^\infty \frac{dx_1}{x_1}\int_{x_1}^\infty \frac{dx_2}{x_2}e^{-2\delta\sqrt{\cdot}x_2}[e^{-2\sqrt{\cdot}x_1}-1]e^{-\beta x_2}, \tag{20}$$

so

$$\begin{aligned}\frac{-dI(\delta,\beta)}{d\beta} &= \int_0^\infty \frac{dx_1}{x_1}\int_{x_1}^\infty dx_2 e^{-2\delta\sqrt{\cdot}x_2}[e^{-2\sqrt{\cdot}x_1}-1]e^{-\beta x_2}\\ &= \int_0^\infty \frac{dx_1}{x_1}\frac{e^{-[2\delta\sqrt{\cdot}+\beta]x_1}}{2\delta\sqrt{\cdot}+\beta}[e^{-2\sqrt{\cdot}x_1}-1]\\ &= \int_0^\infty \frac{dx_1}{2\delta\sqrt{\cdot}+\beta}e^{-[2\delta\sqrt{\cdot}+\beta]x_1}\sum_1^\infty\frac{(-2\sqrt{\cdot}x_1)^{i-1}(-2\sqrt{\cdot})}{i!}\\ &= -\frac{1}{2\delta\sqrt{\cdot}+\beta}\ln\left(1+\frac{2\sqrt{\cdot}}{2\delta\sqrt{\cdot}+\beta}\right).\end{aligned}$$

Set $\beta := 2\sqrt{\cdot}v$,

$$I(\delta,0) = -\int_0^\infty \frac{1}{\delta+v}\ln\left(1+\frac{1}{\delta+v}\right)dv. \tag{21}$$

When $\delta=1$, (A.1) leads to $\frac{-4\sqrt{2}}{\sqrt{\alpha}}\cdot\frac{-\pi^2}{12}$ and after inverting to $\frac{4\pi^2}{6}$ which is correct.

Dimension 3

Equ.(3) leads to

$$3!\frac{2}{\alpha}\int_0^\infty \frac{dx_1}{x_1}\int_{x_1}^\infty \frac{dx_2}{x_2}\int_{x_2}^\infty \frac{dx_3}{x_3}$$
$$\{-2[e^{-2\sqrt{\cdot}(x_1+x_3)}-e^{-2\sqrt{\cdot}x_3}] + \quad 2 \text{ similar terms}\}.$$

Again, let us consider

$$I(\gamma)=\int_0^\infty \frac{dx_1}{x_1}\int_{x_1}^\infty \frac{dx_2}{x_2}\int_{x_2}^\infty \frac{dx_3}{x_3}[e^{-2\sqrt{\cdot}(x_1+x_3)}-e^{-2\sqrt{\cdot}x_3}]e^{-\gamma x_3},$$

so

$$\begin{aligned}\frac{-dI(\gamma)}{d\gamma} &= \int_0^\infty \frac{dx_1}{x_1}\int_{x_1}^\infty \frac{dx_2}{x_2}\int_{x_2}^\infty [\quad]e^{-\gamma x_3}dx_3 \\ &= \int_0^\infty \frac{dx_1}{x_1}\int_{x_1}^\infty \frac{dx_2}{x_2}\frac{e^{-[2\sqrt{\cdot}+\gamma]x_2}}{2\sqrt{\cdot}+\gamma}[e^{-2\sqrt{\cdot}x_1}-1].\end{aligned}$$

Setting $\gamma:=2\sqrt{\cdot}u$, we obtain an expression similar to (20) with $\delta=1+u$. (21) immediately leads to

$$\begin{aligned}&-\int_0^\infty\int_0^\infty \frac{1}{1+u+v}\ln\left(1+\frac{1}{1+u+v}\right)\frac{dudv}{1+u}\\ &=-\int_{u=1}^\infty\int_{v=0}^\infty \frac{1}{u+v}\ln\left(1+\frac{1}{u+v}\right)\frac{dudv}{u}\\ &=-\frac{3}{4}\zeta(3)\quad \text{by (A.3).}\end{aligned}$$

We finally obtain $3!\frac{2}{\alpha}\frac{1}{2}\zeta(3)$ and after inverting, $6\sqrt{2\pi}\zeta(3)$ which is correct.

Again only terms extracted from Thm.1 lead to ζ functions but with different logarithmic integrals. Also they finally lead to the simple expression (18).

5 Conclusion

We have constructed a simple and efficient algorithm to compute the generalized covariances $K(x_1,x_2,\dots,x_d)$. Another challenge would be to derive the cross-moments of any order:

$$K(x_1,i_1,x_2,i_2,\dots,x_d,i_d)=\mathbb{E}[\tau^+(x_1)^{i_1}\tau^+(x_2)^{i_2}\dots\tau^+(x_d)^{i_d}].$$

It appears that the first two derivatives

$$\partial^{i_d}_{\beta_d}\partial^{i_{d-1}}_{\beta_{d-1}}\Theta|_{\beta_d=\beta_{d-1}=0}$$

lead to a linear combination of terms of type

$$D_2(d)^\ell D_4(d)^r/[C_1(d)D_1(d)+C_2(d)D_2(d)]^m.$$

The next derivative $\partial^{i_{d-2}}_{\beta_{d-2}}\Theta$, after all simplifications (and setting $\beta_{d-2}=0$) leads to terms of type

$$H(d-1,s,t):=\frac{D_2(k-1)^s}{[C_1(d-1)D_1(d-1)+C_2(d-1)D_2(d-1)]^t},$$

and this pattern appears in all successive derivatives.

So, we can, in principle, construct (complicated) recurrence equations for $H(\cdot,s,t)$ and recover our K by linear combinations. This is quide tedious and up to now, we couldn't obtain such a simple generating function as in Sec. 3.2 .

Another interesting application is related to a surprising relation discovered by Alili [1] : for any real μ,

$$\left[\frac{ds}{X(s)}\right]^2\stackrel{\mathcal{D}}{\equiv}\mu^2\left[\left[\int_0^1 cotanh(\mu X(s))\right]^2-1\right].$$

We could check this equivalence for the second-order moment, but, up to now, the other moments lead to very complicated sums that we could not simplify.

Acknowledgement

The pertinent comments of the referee led to substantial improvements in the presentation.

References

[1] L. Alili. On some hyperbolic principal values of brownian local times. In M.Yor, editor, *Exponential functionals and principal values related to brownian motion*, pages 131–154. Biblioteca de la revista mathematica Iberoamericana, 1997.

[2] P. Biane and M. Yor. Valeurs principales associées aux temps locaux browniens. *Bulletin de la Société Mathématique de France*, 2e série,111:23–101, 1987.

[3] K.L. Chung. Excursions in brownian motion. *Ark. för Math.*, 14:155–177, 1976.

[4] J.W. Cohen and G. Hooghiemstra. Brownian excursion, the M/M/1 queue and their occupation times. *Mathematics of Operations Research*, 6:608–629, 1981.

[5] M. Drmota and B. Gittenberger. On the profile of random trees. *Random Structures and Algorithms*, 10:421–451, 1997.

[6] P. Flajolet. *Analyse d'algorithmes de manipulation d'arbres et de fichiers.* Université Pierre et Marie Curie, Paris, 1981.

[7] P. Flajolet and G. Louchard. Analytic variations on the Airy distribution. Submitted.

[8] P. Flajolet and A. Odlyzko. Singularity analysis of generating functions. *SIAM Journal on Discrete Mathematics*, 3(2):216–240, 1990.

[9] P. Flajolet, P. Poblete, and A. Viola. On the analysis of linear probing hashing. *Algorithmica*, 22:490–515, 1998.

[10] P. Flajolet and B. Sedgewick. *An Introduction to the Analysis of Algorithms.* Addison-Wesley, 1996.

[11] R.K. Getoor and M.J. Sharpe. Excursions of brownian motion and Bessel processes. *Zeitschrift für Wahrscheinlichkeitstheorie*, 47:83–106, 1979.

[12] B. Gittenberger. On the countour or random trees. *SIAM Journal on Discrete Mathematics*, To appear.

[13] B. Gittenberger and G. Louchard. The brownian excursion multi-dimensional local time density. *Journal of Applied Probability*, 36:350–373, 1999.

[14] G. Hooghiemstra. On the explicit form of the density of brownian excursion local time. *Proceedings of the American Mathematical Society*, 84:127–130, 1982.

[15] F.B. Knight. On the excursion process of brownian motion. *Transactions of the American Mathematical Society*, 258(1):77–86, 1980.

[16] F.B. Knight. On the excursion process of brownian motion. *Zbl. Math.*, 426(abstract 60073), 1980.

[17] L. Lewin. *Dilogarithms and associated functions.* Mac Donald, 1958.

[18] G. Louchard. The brownian excursion area: a numerical analysis. *Computers and Mathematics with Applications*, 10(6):413–417, 1984.

[19] G. Louchard. Kac's formula, Levy's local time and brownian excursion. *Journal of Applied Probability*, 21:479–499, 1984.

[20] G. Louchard. Brownian motion and algorithm complexity. *BIT*, 26:17–34, 1986.

[21] G. Louchard, B. Randrianarimanana, and R. Schott. Dynamic algorithms in D.E. Knuth's model: a probabilistic analysis. *Theoretical Computer Science*, 93:201–225, 1992.

[22] A. Meir and J.W. Moon. On the altitude of nodes in random trees. *Canadian Journal of Mathematics*, 30:997–1015, 1978.

[23] J. Spencer. Enumerating graphs and brownian motion. *Communications on Pure and Applied Mathematics*, L:291–294, 1997.

Appendix

A1

$$\int_1^\infty \frac{du}{u} \ln\left(1+\frac{1}{u}\right) = \int_0^\infty dv \;\ln(1+e^{-v})$$
$$= \sum_{i=1}^\infty \frac{(-1)^{i-1}}{i} \int_0^\infty dv\; e^{-iv} = \sum_{i=1}^\infty \frac{(-1)^{i-1}}{i^2} = \frac{1}{2}\zeta(2).$$

A2

$$\int_1^\infty \int_1^\infty \frac{dudv}{uv} \ln\left(1+\frac{1}{uv}\right) = \frac{3}{4}\zeta(3).$$

The proof is similar to A1's proof.

$$I_1 := \int_1^\infty \int_1^\infty \frac{dudv}{uv} \ln\left(1+\frac{1}{uv+v-1}\right) = \frac{5}{8}\zeta(3).$$

Proof Put $u = 1/u, v = 1/v$,

$$\begin{aligned} I_1 &= \int_0^1 \int_0^1 \frac{dudv}{uv} \ln\left(1+\frac{uv}{1+u-uv}\right) \\ &= \int_0^1 \int_0^1 \frac{dudv}{uv} \ln\left(\frac{1+u}{1+u-uv}\right) \\ &= \int_0^1 \frac{dv}{v} \left\{\int_0^1 \frac{du}{u} \ln(1+u) - \int_0^1 \frac{du}{u} \ln[1+u(1-v)]\right\} \\ &= \int_0^1 \frac{dv}{v} \left\{\int_0^1 \frac{du}{u} \ln(1+u) - \int_0^{1-v} \frac{dw}{w} \ln(1+w)\right\} \\ &= \int_0^1 \frac{dv}{v} \int_{1-v}^1 \frac{du}{u} \ln(1+u). \end{aligned}$$

By parts, with $f = \int_{1-v}^1 \frac{du}{u} \ln(1+u), g = \frac{1}{v}$, we derive

$$I_1 = \ln\ v \int_{1-v}^1 \frac{du}{u} \ln(1+u)|_{v=0}^{v=1} - \int_0^1 dv\;\ln\ v \cdot \frac{\ln(2-v)}{1-v}.$$

Finally put $v = 1-x$, we have

$$I_1 = -\int_0^1 \frac{dx}{x} \ln(1-x)\ln(1+x).$$

Let us now prove a few lemmas.
Lemma $J_1 := \int_0^1 \frac{dx}{x} \ln^2(1-x) = 2\zeta(3)$.

Proof Put $x = 1 - y$,

$$\begin{aligned} J_1 &= \int_0^1 dy \frac{\ln^2 y}{1-y} = \int_0^\infty du \frac{e^{-u}}{1-e^{-u}} u^2 \\ &= \int_0^\infty du\ u^2 \frac{1}{e^u - 1} = \sum_{p=1}^\infty \int_0^\infty du\ u^2 e^{-pu} = 2\zeta(3). \end{aligned}$$

■

Lemma $J_2 := \int_0^1 dx \frac{\ln^2(1+x)}{x} = 1/4\zeta(3)$.

Proof Let $C = \{e^{i\theta}, \theta = 0 \ldots \pi\} \bigcup [-1+1], \int_c dz \frac{\ln^2(1+z)}{z} = 0$ (as integrand is regular at $z = 0$).

$$\int_{-1}^0 \frac{dx}{x} \ln^2(1+x) + \int_0^1 \frac{dx}{x} \ln^2(1+x) + i \int_0^\pi d\theta\ \ln^2(1+e^{i\theta}) = 0,$$

$$-\int_0^1 \frac{dy}{y} \ln^2(1-y) + \int_0^1 \frac{dx}{x} \ln^2(1+x) + i \int_0^\pi d\theta\ \ln^2(1+\cos\theta + isin\theta) = 0,$$

$$\begin{aligned} \int_0^1 \frac{dx}{x} \ln^2(1+x) &= 2\zeta(3) + \Im \int_0^\pi d\theta \left\{ \frac{1}{2} \ln[(1+\cos\theta)^2 + \sin^2\theta] + i \tan^{-1} \frac{\sin\theta}{1+\cos\theta} \right\}^2 \\ &= 2\zeta(3) + \int_0^\pi d\theta\ \theta[\ln 2 + \ln\cos(\theta/2)]. \end{aligned}$$

As $\ln|\cos x| = -\ln 2 + \cos 2x - \frac{\cos 4x}{2} + \frac{\cos 6x}{3} \cdots$

$$\begin{aligned} \int_0^1 \frac{dx}{x} \ln^2(1+x) &= 2\zeta(3) + \sum_{n=1}^\infty \frac{(-)^{n-1}}{n} \int_0^\pi d\theta\ \theta \cos n\theta \\ &= 2\zeta(3) - 2 \sum_{\substack{n=1 \\ \text{(odd)}}}^\infty 1/n^3 = 2\zeta(3) - 2(\frac{7}{8}\zeta(3)) \\ &= 1/4\zeta(3). \end{aligned}$$

■

Finally let

$$\begin{aligned} K &:= \int_0^1 \frac{dx}{x} [\ln(1+x) + \ln(1-x)]^2 \\ &= \int_0^1 \frac{dx}{x} \ln^2(1+x) + \int_0^1 \frac{dx}{x} \ln^2(1-x) + 2\int_0^1 \frac{dx}{x} \ln(1+x) \cdot \ln(1-x) \\ &= 1/4\zeta(3) + 2\zeta(3) + 2\int_0^1 \frac{dx}{x} \ln(1+x) \cdot \ln(1-x). \end{aligned}$$

But

$$K = \int_0^1 \frac{dx}{x} \ln^2(1-x^2) = \frac{1}{2} \int_0^1 \frac{dy}{y} \ln^2(1-y) = \zeta(3).$$

So

$$\begin{aligned} I_1 &= -\int_0^1 \frac{dx}{x} \ln(1+x) \cdot \ln(1-x) = \frac{1}{2}[\frac{1}{4}\zeta(3) + 2\zeta(3) - \zeta(3)] \\ &= \frac{5}{8}\zeta(3). \end{aligned}$$

■

$$I_2 := \int_1^\infty \int_1^\infty \frac{dudv}{uv} \ln\left(1 + \frac{1}{uv+v}\right) = \frac{13}{24}\zeta(3).$$

Proof Put $u = 1/u, v = 1/v$,

$$I_2 = -\int_0^1 \frac{dv}{1+v} \ln\ v \cdot \ln(2+v).$$

(calculation similar to previous case)

The quickest way to obtain the value of this integral is apparently to connect it to the trilogarithm integral, defined as

$$Li_3(z) = \int_0^z \frac{dz}{z} Li_2(z),$$

where $Li_2(z)$ is the dilogarithm integral

$$Li_2(z) = -\int_0^z \frac{dz}{z} \ln(1-z),$$

(see for instance Lewin [17]). Among the many formulae we find on p. 273, [3.5.1]

$$\begin{aligned}
K(c,x) \quad &:= \quad \int_0^x \frac{dy}{y} \ln(1-y) \cdot \ln(1-cy) \\
&= \quad Li_3\left(\frac{1-cx}{1-x}\right) - Li_3\left(\frac{1-cx}{c(1-x)}\right) - Li_3(1-x) - Li_3(1-cx) + Li_3(1/c) + Li_3(1) \\
&+ \quad \ln(1-cx)[Li_2(1/c) - Li_2(x)] \\
&+ \quad \ln(1-x)[Li_2(1-cx) - Li_2(1/c) + \pi^2/6] \\
&+ \quad 1/2 \ln\ c \cdot \ln^2(1-x).
\end{aligned}$$

[The proof is long, and to quote the author: "... There does not appear to be any obvious way of reaching this quite compact (sic) result in a more direct manner, through it is a simple (sic) matter to verify it by differentiation..."]

We derive successively

$$\begin{aligned}
I_2 \quad &= \quad -\int_0^1 \frac{dv}{1+v} \ln v \cdot \ln(2+v) \\
(\text{put } v = y-1) \quad &= \quad -\int_1^2 \frac{dy}{y} \ln(y-1) \cdot \ln(y+1) \\
(y = -u) \quad &= \quad -\int_{-1}^{-2} \frac{du}{u} \ln(-u-1) \cdot \ln(-u+1) \\
&= \quad -\int_{-1}^{-2} \frac{du}{u} \ln(1-u) \cdot \Re \ln(1+u) \\
&= \quad -\Re[K(-1,-2) - K(-1,-1)] \\
&= \quad \Re\left[-Li_3(2) + \ln 2[-Li_2(-1) + \pi^2/6]\right. \\
&- \quad \left. Li_3(-1/3) + Li_3(1/3) + Li_3(3) + Li_3(-1) - \ln 3 \cdot \pi^2/6\right].
\end{aligned}$$

Making extensive use of properties of the dilogarithm and trilogarithms in particular $Li_2(-1) = -\pi^2/12; Li_3(1) = \zeta(3)$

$$Li_3\left(\frac{1-x}{1+x}\right) - Li_3\left(\frac{x-1}{x+1}\right) = 2Li_3(1-x) + 2Li_3\left(\frac{1}{1+x}\right) - 1/2 Li_3(1-x^2)$$
$$-\pi/4 Li_3(1) + \pi^2/6 \ln(1+x) - 1/3 \ln^3(1+x),$$

shows that

$$\begin{aligned}
\Re[Li_3(3) - Li_3(1/3)] &= \pi^2/3\ln 3 - 1/6\ln^3 3, \\
Li_3(-1/3) - 2Li_3(1/3) &= -13/6\zeta(3) + \pi^2/6\ln 3 - 1/6\ln^3 3, \\
\Re Li_3(2) &= 7/8\zeta(3) + \pi^2/4\ln 2,
\end{aligned}$$

and so

$$I_2 = 13/24\zeta(3).$$

■

A3

$$\begin{aligned}
\int_{u=1}^{\infty}\int_{v=0}^{\infty}\frac{dudv}{u(u+v)}\ln\left(1+\frac{1}{u+v}\right) &= \frac{3}{4}\zeta(3), \\
\int_{u=1}^{\infty}\int_{v=0}^{\infty}\frac{dudv}{u(1+u+v)}\ln\left(1+\frac{1}{1+u+v}\right) &= \frac{13}{24}\zeta(3), \\
\int_{u=1}^{\infty}\int_{v=0}^{\infty}\frac{dudv}{u(1+u+v)}\ln\left(1+\frac{1}{u+v}\right) &= \frac{5}{8}\zeta(3).
\end{aligned}$$

Proofs are similar to A2's proofs.