Analysis of a Recurrence Related to Critical Nonhomogeneous Branching Processes

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Abstract

Some classes of controlled branching processes (with nonhomogeneous migration or with nonhomogeneous state-dependent immigration) lead in the critical case to a recurrence for the extinction probabilities. Under some additional conditions it is known that this recurrence depends on some parameter β and converges for $0 < \beta < 1$. Now we show that the above recurrence does converge for all positive values of the parameter β which leads to an extension of some limit theorems for the corresponding branching processes. We also give a generalization of the recurrence and an asymptotic analysis of its behaviour.

1 Introduction

Branching processes can be interpreted as mathematical models of population dynamics where the reproduction of the individuals follows some stochastic laws.

It is well known that the history of branching processes begins with the pioneer papers of Bienaymé (1845) and Galton and Watson (1874) analysing some demographic problems. However the terminology "branching processes" was introduced by A.N.Kolmogorov and among the first asymptotic results of the modern mathematical basis were obtained by him and his students (see Kolmogorov [12], Kolmogorov and Dmitriev [13], Kolmogorov and Sevastyanov [14]). More details of the further development can be obtained from the books of Harris [10], Sevastyanov [17], Athreya and Ney [2], Jagers [11], Asmussen and Hering [1] and Guttorp [9].

The classical Bienaymé-Galton-Watson branching process can be used to describe the growth of an isolated population with an independent reproduction of the individuals. However, in real situations there is always an interaction with the environment and/or between the individuals, which means that, in general, the evolutions of the individuals are not independent.

Controlled branching processes represent one approach in describing this more complicated situation.

Yanev [21] considered a generalization of the classical BGW branching process in the case of *random control functions* (RCF):

$$Z_{n+1} = \sum_{j=1}^{\varphi_n(Z_n)} X_n(j),\ n = 0, 1, 2, \ldots, \tag{1}$$

where $\varphi_n = \{\varphi_n(k), k = 0, 1, 2, \ldots\}, n \geq 0$, are independent non-negative integer-valued random processes with identical one-dimensional distributions.

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Note that the classical BGW branching process follows by this definition if $\varphi_n(k) \equiv k$ a.s.

First Sevastyanov and Zubkov [18] obtained conditions for extinction or non-extinction in the case $\varphi_n(k) \equiv \varphi(k)$ a.s., where $\varphi(k)$ is a deterministic function and $\varphi(n) \sim \alpha n, n \to \infty, \alpha > 0$.

Conditions for extinction or non-extinction for the process (1)were obtained by Yanev [21] [22] and Bruss [4] assuming a.s. linear growth of the RCF. Yanev [22] assumed also a random environment. Gonzales, Molina and Del Puerto [6] [7] investigated the model (1) when the means of the RCF have a linear growth.

The controlled branching processes with multi-type RCF can be defined as follows:

$$Z_{n+1} = \sum_{i \in I} \sum_{j=1}^{\varphi_{i,n}(Z_n)} X_{i,n}(j),\ n = 0, 1, 2, \dots, \tag{2}$$

where I is an index set (finite or infinite) and the set $X = \{X_{i,n}(j)\}$ consists of i.i.d. for each fixed i random variables. On the other hand, the set X is independent of the set of the RCF $\varphi = \{\varphi_{i,n}(k)\}$, where $\varphi_n = \{\varphi_{i,n}(k), i \in I, k = 0, 1, 2, \dots\}$ are independent random fields with identical one-dimensional distributions.

Zubkov [24] investigated the case of deterministic control functions, i.e. $\varphi_{i,n}(k) \equiv \varphi_i(k)$ a.s. For more general controls, sufficient extinction criteria were obtained by Bruss [3]. Yanev and Yanev [19] [20] obtained conditions for extinction or non-extinction in the general model (2) with random environment.

Note that the general model (2) describes a very large class of stochastic processes, in particular all Markov chains and the most popular considered models of branching processes in discrete time: BGW process, processes with immigration, branching processes with random migration components, population-size-dependent branching processes, branching processes in random environment and so on. However, in all these models it is always fulfilled the condition $\sum_{i \in I} \varphi_{i,n}(k) \to \infty, a.s., n \to \infty$, which can be considered as a general branching property.

Of course, in the general situation (2) it is not possible to obtain detailed asymptotic results. That is why it is very interesting to consider every particular case where such asymptotic analysis is possible. On the other hand, it is also very important for applications.

Yanev and Mitov [23] considered the following model allowing nonhomogeneous migration: in the general definition (2) it is assumed that $I = \{1, 2\}$ and

$$\varphi_{1,n}(k) = \max\{\min(k, k + Y_n), 0\}, \varphi_{2,n}(k) = \max\{Y_n, 0\}, \tag{3}$$

where the independent r.v. Y_n have the following distributions

$$P(Y_n = -1) = p_n, P(Y_n = 0) = q_n, P(Y_n = 1) = r_n,\ p_n + q_n + r_n = 1,\ n = 0, 1, 2, \dots \tag{4}$$

Then by (2)-(4) it follows that $\{Z_n\}$ is a nonhomogeneous Markov chain which can be represented as follows:

$$Z_{n+1} = \sum_{j=1}^{Z_n} X_{1,n}(j) + M_n,\ n = 0, 1, 2, \dots \tag{5}$$

where [1]

$$\begin{aligned} M_n &= -X_{1,n}(1)[\![Z_n > 0]\!] \text{ with probability } p_n, \\ &= 0 \text{ with probability } q_n, \\ &= X_{2,n}(1) \text{ with probability } r_n. \end{aligned}$$

One can assume without loss of generality that $Z_0 = 0$ a.s.

[1]Here we use the indicator function notation proposed by Knuth et al. [8].

The process (5) allows for the following interpretation. In the $(n+1)th$ generation $(n = 0, 1, 2, \dots)$ the next three situations are possible: (i) with probability p_n, one family is eliminated and does not take part in a further evolution (emigration); (ii) with probability q_n, there is no migration (like in a BGW process); (iii) with probability r_n, there is an immigration of new individuals according to a p.g.f. $G(s) = E(s^{X_{2,n}(1)})$.

Let $F(s) = E(s^{X_{1,n}(1)})$ be the offspring p.g.f. and $H_n(s) = E(s^{Z_n})$ be the p.g.f. of the process (5). Then by (5) one can obtain the equation (see Yanev and Mitov [23]):

$$H_{n+1}(s) = U_n(n,s) + \sum_{k=0}^{n} p_{n-k} H_{n-k}(0)[1 - 1/F_{k+1}(s)]U_{k-1}(n,s), \tag{6}$$

where $U_{k-1}(n,s) = \prod_{i=0}^{k} \Delta_{n-i}(F_i(s))$, $\Delta_n(s) = p_n/F(s) + q_n + r_n G(s)$ and $F_i(s)$ is the ith iterate of $F(s)$, i.e. $F_0(s) = s$, $F_{i+1}(s) = F(F_i(s))$, $i = 0, 1, 2, \dots$.

Further on the critical case $F'(1) = 1, 0 < F''(1) = 2\alpha < \infty$, with a balanced migration component $r_n EX_{2,n}(1) \equiv p_n \sim K \log n$ is considered. In this case it follows by (6):

$$H_{n+1}(0) = 1 - \sum_{k=0}^{n} (1 - F_k(0)) p_{n-k} H_{n-k}(0) + O(1/\log n), \tag{7}$$

where $1 - F_n(0) \sim 1/(\alpha n)$, $n \to \infty$. This corresponds to the famous asymptotic result of Kolmogorov [12] for the probability of non-extinction in a BGW process.

When $\beta = K/\alpha < 1$ then it was shown in Yanev and Mitov [23] that $\lim H_n(0) = 1/(1+\beta)$, respectively $\lim R_n = \beta/(1+\beta)$, where $R_n = 1 - H_n(0)$ is the probability of non-extinction.

The case $\beta \geq 1$ remained open for a very long time.

In this paper, we will first consider recurrences of the above type:

Main recurrence equation

$$a_{n+1} = T_n - \sum_{k=0}^{n} a_k P_k Q_{n-k}, \, (n \geq 0), \tag{8}$$

where P_n, Q_n and T_n are given non-negative sequences. We establish general properties of the solutions a_n in Section 2. It will turn out that, under quite natural monotonicity conditions, all solutions of (8) are bounded and that there are solutions with $0 \leq a_n \leq 1$.

Finally, in Section 3 we discuss the case $P_n \sim \beta/\log n, Q_n \sim 1/n$ and $T_n \to 1$ (that is, the original problem from Yanev and Mitov [23]) in more details and we show that a_n is convergent for any choice of $\beta > 0$. This result allows us to generalize Theorem 1 of Yanev and Mitov [23].

Theorem 1.1 *Let $F'(1) = 1, 0 < F''(1) = 2\alpha < \infty$ and $r_n EX_{2,n}(1) \equiv p_n \sim K/\log n$, $n \to \infty$, $K > 0$. Then for all $\beta = K/\alpha$ we have:*

$$\lim_{n\to\infty} P(Z_n > 0) = \beta/(1+\beta), \; E(Z_n) \sim \beta\alpha n/\log n, \; E(Z_n^2) \sim \beta\alpha^2 n^2/\log n$$

and

$$\lim_{n\to\infty} P\left(\log Z_n/\log n \leq x \,|\, Z_n > 0\right) = x, \; 0 \leq x \leq 1.$$

Branching processes with decreasing state-dependent immigration represent another example where the same analytical problem appears.

Let us assume in (2) that $I = \{1, 2\}, \varphi_{1,n}(k) \equiv k$ and $\varphi_{2,k}(k) \equiv \delta_{k,0}$, where as usual $\delta_{k,0} = 0$ for $k \geq 1$ and $\delta_{0,0} = 1$. Then

$$Z_{n+1} = \sum_{j=1}^{Z_n} X_{1,n}(j) + X_{2,n}(1)[\![Z_n = 0]\!], \; n = 0, 1, 2, \dots, \tag{9}$$

where $F(s) = E(s^{X_{1,n}(1)})$ is the offspring p.g.f.

The critical case $F'(1) = 1, 0 < F''(1) = 2\alpha < \infty$ is considered in Mitov and Yanev [16] and an equation similar to (7) is obtained. It is also proved that $\lim_{n\to\infty} P(Z_n > 0) = \beta/(1+\beta)$ for $\beta = K/\alpha < 1$ and it is pointed out that the case $\beta \ge 1$ is an open problem (see Remark, p.30).

Therefore this problem is now solved and we are able to prove a similar limit theorem as Theorem 1.1, which generalizes the corresponding Theorem 1 from Mitov and Yanev [16].

Finally, the continuous time analogy of the classical branching process is considered by Mitov, Vatutin and Yanev [15], where it is also pointed out that the case $\beta \ge 1$ is "still an open problem" (Remark, p.707).

Section 4 concludes the paper.

2 Boundedness and Convergence

2.1 Convergence by "Contraction"

If one wants to obtain convergent solutions a_n of (8). then one way to attack this problem is to first assume that the sequences T_n and

$$C_n := \sum_{k=0}^{n} P_k Q_{n-k} \tag{10}$$

are convergent, say $T = \lim_{n\to\infty} T_n$ and $C = \lim_{n\to\infty} C_n$. Then the *expected limit* α of the sequence a_n is

$$\alpha = \frac{T}{C+1}.$$

Our first result covers the case $C < 1$ that is also related to the case covered in [23]. The constant $C < 1$ serves (more or less) as a contraction factor that forces the sequence being convergent.

Theorem 2.1 *Suppose that* P_n, Q_n, T_n *are non-negative sequences such that* $\lim_{n\to\infty} T_n = T$, $\lim_{n\to\infty} Q_n = 0$, *and* $\lim_{n\to\infty} C_n = C < 1$. *Then every sequence* a_n *that satisfies (8) is convergent with limit*

$$\lim_{n\to\infty} a_n = \frac{T}{C+1}.$$

Proof. First of all, it is easy to show that the sequence a_n is bounded. Set $\overline{T} = \sup_{n\ge 0} T_n$ and $\overline{C} = \sup_{n\ge 0} C_n$. First, suppose that $\overline{C} < 1$. Then it follows by induction that

$$|a_n| \le \overline{C}^n |a_0| + (\overline{C}^{n-1} + \cdots + C + 1)\overline{T}$$

and, thus, a_n is bounded. (If $\overline{C} \ge 1$ we just have to shift the argument to $n \ge n_0$, where n_0 is chosen in a way that $\sup_{n\ge n_0} C_n < 1$.)

Furthermore, since C_n is bounded and $C_n \ge P_{n-\ell_0} Q_{\ell_0}$ (where $Q_{\ell_0} \ne 0$) it follows that P_n is bounded, too.

Next set $b_n = a_n - \frac{T}{C+1}$. Then b_n satisfies the recurrence

$$b_{n+1} = -\sum_{k=0}^{n} b_k P_k Q_{n-k} + \tilde{T}_n,$$

where

$$\tilde{T}_n = T_n - (C_n + 1)\frac{T}{C+1} \to 0.$$

We again suppose that $\overline{C} = \sup_{n\ge 0} C_n < 1$ (otherwise we have to *shift* the argument as above). We further set $\overline{B} = \sup_{n\ge 1} |b_n|$ and $\overline{P} = \sup_{n\ge 1} |P_n|$. Since $Q_n \to 0$ there exists an increasing integer sequence n_k (with $n_0 = 0$) and a sequence $\eta_k > 0$ that is monotonely decreasing to 0 such that

$$Q_n + Q_{n-1} + \cdots + Q_{n-n_k} \le \eta_k$$

for all $n \geq n_{k+1}$. Set

$$B_k := \max_{n_k \leq n < n_{k+1}} |b_n|$$

and

$$\tau_k := \overline{B}\,\overline{P}\,\eta_k + \sup_{n \geq n_{k+1}} |\tilde{T}_n|.$$

By construction we have $\tau_k \to 0$. Finally choose $\tilde{B}_0 \geq B_0$ such that $\tilde{B}_1 = \overline{C}\tilde{B}_0 + \tau_0 \leq \tilde{B}_0$ and set

$$\tilde{B}_{k+1} = \overline{C}\tilde{B}_k + \tau_k$$

for $k \geq 0$. Then it follows by induction that $\tilde{B}_{k+1} \leq \tilde{B}_k$ and that

$$\tilde{B}_k = \overline{C}^k \tilde{B}_0 + \sum_{\ell=0}^{k-1} \overline{C}^{k-1-\ell} \tau_\ell \to 0$$

as $k \to \infty$, that is, $\tilde{B}_k$ converges monotonely to 0.

In a final step we show by induction that $B_k \leq \tilde{B}_k$ for all $k \geq 0$. Of course this shows that $b_n \to 0$ and consequently $a_n \to T/(C+1)$.

By definition we have $B_0 \leq \tilde{B}_0$. Now suppose that $B_k \leq \tilde{B}_k$ for some $k \geq 0$. Then, for $n = n_{k+1}$ we have

$$\begin{aligned}
|b_{n_{k+1}}| &\leq \sum_{k=0}^{n_k} |b_k| P_k Q_{n_{k+1}-k} + \sum_{k=n_k+1}^{n_{k+1}-1} |b_k| P_k Q_{n-k} + |\tilde{T}_{n-1}| \\
&\leq \overline{B}\,\overline{P}\,\eta_k + \overline{C} B_k + |\tilde{T}_{n-1}| \\
&\leq \overline{C}\tilde{B}_k + \tau_k \\
&\leq \tilde{B}_{k+1}
\end{aligned}$$

Note that we also have $|b_{n_{k+1}}| \leq \tilde{B}_k$. In the same way we get (inductively) for $n_{k+1} \leq n < n_{k+2}$

$$|b_n| \leq \overline{C}\tilde{B}_k + \tau_k = \tilde{B}_{k+1}$$

and, thus,

$$B_{k+1} = \max_{n_{k+1} \leq n < n_{k+2}} |b_n| \leq \tilde{B}_{k+1}.$$

This completes the proof of Theorem 2.1. ■

Remark 2.2 *Note that with help of the same proof-techniques it follows that a_n is bounded if*

$$\limsup_{n\to\infty} T_n < \infty \quad \textit{and} \quad \limsup_{n\to\infty} C_n < 1$$

and that $a_n \to 0$ if

$$\lim_{n\to\infty} T_n = 0 \quad \textit{and} \quad \limsup_{n\to\infty} C_n < 1$$

2.2 Boundedness Conditions

We are now interested in conditions on the sequences P_n, Q_n, and T_n that imply that all solutions a_n of (8) are bounded regardless of $C < 1$ or not so that we cannot use a *contraction argument*, like in the preceding Remark.

Theorem 2.3 *Suppose that Q_n and C_n are monotone convergent sequences, $Q_0 P_n \le 1$ for $n \ge n_0$ (for some n_0) and*

$$\sum_{n\ge 0} |T_{n+1} - T_n| < \infty.$$

Then every sequence a_n that satisfies (8) is bounded.

Proof. Since C_n is bounded it follows as in the proof of Theorem 2.1 that P_n and Q_n have to bounded, too.

First, consider the case when Q_n is increasing (to a finite limit Q). Since C_n is convergent, it follows that

$$\sum_{n\ge 0} P_n < \infty.$$

Now, by subtracting (8) for a_{n+2} and a_{n+1} it follows that a_n satisfies the recurrence

$$a_{n+2} = a_{n+1}(1 - P_{n+1}Q_0) + \sum_{k=0}^{n} a_k P_k (Q_{n-k} - Q_{n-k+1}) + T_{n+1} - T_n,$$

too. Set $A_n = \max_{0\le k\le n} |a_k|$. Then we have

$$\begin{aligned}
|a_{n+2}| &\le A_{n+1}\left(1 - P_{n+1}Q_0 - \sum_{k=0}^{n} a_k P_k (Q_{n-k} - Q_{n-k+1})\right) + |T_{n+1} - T_n| \\
&= A_{n+1}\left(1 + (C_{n+1} - C_n) - 2P_{n+1}Q_0\right) + |T_{n+1} - T_n|
\end{aligned}$$

If Q_n is increasing we certainly have

$$C_{n+1} - C_n = \sum_{k=0}^{n} a_k P_k (Q_{n+1-k} - Q_{n-k}) + P_{n+1}Q_0 \ge 0,$$

(that is, it is not necessary to assume that C_n is monotone in that case). Thus, the series

$$\sum_{n\ge 0} (C_{n+1} - C_n - 2P_{n+1}Q_0)$$

is absolutely convergent and from

$$A_{n+2} \le A_{n+1}\left(1 + (C_{n+1} - C_n) - 2P_{n+1}Q_0\right) + |T_{n+1} - T_n|$$

it follows that A_n is a bounded sequence. Thus, a_n is bounded, too.

If Q_n is decreasing we get (as above)

$$\begin{aligned}
|a_{n+2}| &\le A_{n+1}\left(1 - P_{n+1}Q_0 + \sum_{k=0}^{n} a_k P_k (Q_{n-k} - Q_{n-k+1})\right) + |T_{n+1} - T_n| \\
&= A_{n+1}\left(1 - (C_{n+1} - C_n)\right) + |T_{n+1} - T_n|.
\end{aligned}$$

Hence, if C_n is increasing then we have

$$A_{n+2} \le A_{n+1} + |T_{n+1} - T_n|$$

and if C_n is decreasing

$$A_{n+2} \le A_{n+1}\left(1 + (C_n - C_{n+1})\right) + |T_{n+1} - T_n|.$$

In both cases, it follows that A_n is bounded. ■

Let us consider the example $T_n = 1, P_n = \beta/\log n$ (for $n \geq 2$, $P_0 = P_1 = 0$) and $Q_n = 1/(n+1)$. Then

$$\begin{aligned} C_n &= \beta \sum_{k=2}^{n} \frac{1}{(n-k+1)\log k} \\ &= \beta\left(1 + \frac{\gamma}{\log n} + O\left(\frac{1}{(\log n)^2}\right)\right) \\ &\to \beta, \end{aligned}$$

where γ is the Euler's constant. Furthermore, after some algebra one obtains

$$\begin{aligned} C_n &= \beta\left\{\frac{1}{\log n}[\log n + \gamma] + \frac{\pi^2}{6\log^2 n}\right\} + O\left(\frac{1}{\log^3 n}\right) \\ C_n - C_{n+1} &= \beta\frac{\gamma}{n(\log n)^2} + O\left(\frac{1}{n(\log n)^3}\right) > 0 \end{aligned}$$

for sufficiently large n. Thus, we can apply Theorem 2.3 and obtain that every solution a_n is bounded.

Next we present a second condition for boundedness that does not rely on monotonicity or convergence properties of C_n. It also shows that there exist solutions with $0 \leq a_n \leq 1$, that is, a_n may be interpreted as probabilities.

Theorem 2.4 *Suppose that P_n and Q_n are non-negative and strictly monotonely decreasing sequences, $T_n = 1$, and $P_3Q_0 \leq 1$. Then there exists a solution a_n of (8) with $0 \leq a_n \leq 1$ for all $n \geq 0$. Furthermore, all solutions of (8) are bounded.*

Proof. The idea of the proof is to show that there exist two different solutions $a_n^{(1)}$, $a_n^{(2)}$ with $0 \leq a_n^{(1)}, a_n^{(2)} \leq 1$. Then any solution is given by

$$a_n = a_n^{(1)} + \frac{a_0 - a_0^{(1)}}{a_0^{(2)} - a_0^{(1)}}(a_n^{(2)} - a_n^{(1)})$$

and is, thus, bounded, too.

We consider several cases. First, suppose that $P_1Q_0 \leq 1$ and consider the system of equations

$$\begin{aligned} a_0P_0Q_0 + a_1 &= 1, \\ a_0P_0Q_1 + a_1P_1Q_0 + a_2 &= 1. \end{aligned}$$

Since $P_0Q_0 < P_0Q_1$ and $1 \geq P_1Q_0$ it follows that for every a_2 with $0 \leq a_2 \leq 1 - Q_1/Q_0$ there is a unique solution for a_0 and a_1 with $0 \leq a_0, a_1 \leq 1$. Next set recursively

$$a_{n+1} = 1 - \sum_{k=0}^{n} a_kP_kQ_{n-k} \quad (n \geq 2).$$

We induction we show that we always have $0 \leq a_{n+1} \leq 1$. Note that $P_1Q_0 \leq 1$ also implies $P_nQ_0 \leq 1$ for all $n \geq 1$. Assume that $0 \leq a_0, a_1, \ldots, a_n \leq 1$ for some $n \geq 2$. Then

$$0 \leq \sum_{k=0}^{n} a_kP_kQ_{n-k} \leq \sum_{k=0}^{n-1} a_kP_kQ_{n-1-k} + a_n = 1.$$

and consequently $0 \leq a_{n+1} \leq 1$. Thus, we have $0 \leq a_n \leq 1$ for all $n \geq 0$.

Now we use this construction for $a_2^{(1)} = 0$ and for $a_2^{(2)} = 1 - Q_1/Q_0 > 0$. Consequently we obtain two different solutions $a_n^{(1)}$, $a_n^{(2)}$ with $0 \leq a_n^{(1)}, a_n^{(2)} \leq 1$ and the result follows.

Second, suppose that $P_1Q_0 > 1$ and $P_2Q_0 \le 1$. Here we consider the system of equations

$$\begin{aligned} a_0P_0Q_0 + a_1 &= 1, \\ a_0P_0Q_1 + a_1P_1Q_0 + a_2 &= 1, \\ a_0P_0Q_2 + a_1P_1Q_1 + a_2P_2Q_0 + a_3 &= 1. \end{aligned} \tag{11}$$

After eliminating a_0 we get

$$a_1 \frac{P_1Q_0 - \frac{Q_1}{Q_0}}{1 - \frac{Q_1}{Q_0}} + a_2 \frac{1}{1 - \frac{Q_1}{Q_0}} = 1, \tag{12}$$

$$a_1 \frac{P_1Q_1 - \frac{Q_2}{Q_0}}{1 - \frac{Q_2}{Q_0}} + a_2 \frac{P_2Q_0}{1 - \frac{Q_2}{Q_0}} + a_3 \frac{1}{1 - \frac{Q_2}{Q_0}} = 1. \tag{13}$$

Note that the coefficients of a_1 satisfy

$$\frac{P_1Q_0 - \frac{Q_1}{Q_0}}{1 - \frac{Q_1}{Q_0}} > \frac{P_1Q_1 - \frac{Q_2}{Q_0}}{1 - \frac{Q_2}{Q_0}} > 0.$$

If

$$\frac{1}{1 - \frac{Q_1}{Q_0}} \ge \frac{P_2Q_0}{1 - \frac{Q_2}{Q_0}} \tag{14}$$

then for arbitrary $a_1, a_2 \ge 0$ that satisfy (12) we obtain

$$0 \le a_1 \frac{P_1Q_1 - \frac{Q_2}{Q_0}}{1 - \frac{Q_2}{Q_0}} + a_2 \frac{P_2Q_0}{1 - \frac{Q_2}{Q_0}} \le a_1 \frac{P_1Q_0 - \frac{Q_1}{Q_0}}{1 - \frac{Q_1}{Q_0}} + a_2 \frac{1}{1 - \frac{Q_1}{Q_0}} = 1$$

and consequently (from (13))

$$0 \le a_3 \le 1 - \frac{Q_2}{Q_0}.$$

Note that $a_1, a_2 \ge 0$ that satisfy (12) are bounded by

$$a_1 \le \frac{1 - \frac{Q_1}{Q_0}}{P_1Q_0 - \frac{Q_1}{Q_0}} \le 1 \quad \text{and} \quad a_2 \le 1 - \frac{Q_1}{Q_0} \le 1.$$

Next set

$$a_0 = \frac{1 - a_1}{1 - P_0Q_0}.$$

Thus, we have found solutions $0 \le a_0, a_1, a_2 \le 1$ of the system (11). Since $P_nQ_0 \le 1$ for all $n \ge 2$ we can proceed as above and obtain $0 \le a_n \le 1$ for all $n \ge 0$. Since $a_1, a_2 \ge 0$ can be chosen in an arbitrary way we again get two different solutions.

Next, consider the case, where (14) is not satisfied, that is,

$$\frac{1}{1 - \frac{Q_1}{Q_0}} < \frac{P_2Q_0}{1 - \frac{Q_2}{Q_0}} \tag{15}$$

Then for every a_3 with

$$0 \le a_3 \le \left(1 - \frac{\frac{P_1Q_1 - Q_2/Q_0}{1 - Q_2/Q_0}}{\frac{P_1Q_0 - Q_1/Q_0}{1 - Q_1/Q_0}}\right)\left(1 - \frac{Q_2}{Q_0}\right)$$

there (uniquely) exist $a_1, a_2 \ge 0$ that satisfy (12) and (13). Setting $a_0 = (1 - a_1)/(1 - P_0Q_0)$ we get a solution of (11) with $0 \le a_0, a_1, a_2 \le 1$ and proceed as in the previous case.

Finally, let us consider the case $P_2Q_0 > 1$ and $P_3Q_0 \le 1$. Here we consider again the system (12), (13) and oberve that

$$\frac{P_1Q_0 - \frac{Q_1}{Q_0}}{1-\frac{Q_1}{Q_0}} > \frac{P_1Q_1 - \frac{Q_2}{Q_0}}{1-\frac{Q_2}{Q_0}} > 0$$

and

$$\frac{1}{1-\frac{Q_1}{Q_0}} \ge \frac{P_2Q_0}{1-\frac{Q_2}{Q_0}}.$$

Thus, we can proceed as the case where $P_1Q_0 > 1$, $P_2Q_0 \le 1$, and (14) are satisfied. This completes the proof of Theorem 2.4. ■

It is not clear whether the above arguments (recursive elimination etc.) can be extended to all cases where P_n and Q_n are monotonely decreasing. The condition $P_3Q_0 \le 1$ seems to be artificial. We leave this as an open problem.

2.3 Convergence Conditions

If $C = \lim C_n \ge 1$ then one would not expect convergence. However, there is a simple trick to reduce the original problem to the contraction case that has been already considered in Theorem 2.1.

Theorem 2.5 *Suppose that P_n, Q_n, T_n are non-negative sequences such that P_n and Q_n converges to 0, $\lim_{n\to\infty} T_n = T$, and $\lim_{n\to\infty} C_n = C \ge 1$. Let $Q_{n,k}^{(r)}$ be recursively defined by $Q_{n,k}^{(1)} = Q_{n-k}$ and*

$$Q_{n,k}^{(r+1)} := \sum_{\ell=k}^{n-r} P_{\ell+1} Q_{\ell-k} Q_{n,\ell+1}^{(r)} \quad (r \ge 1).$$

Suppose that there exists $r \ge 2$ and positive real numbers $c_1, c_2, \ldots, c_r$ with $c_1 + c_2 + \cdots + c_r = 1$ such that

$$\limsup_{n\to\infty} \sum_{k=0}^{n-r+1} P_k \left| \sum_{j=1}^{r} (-1)^j c_j Q_{n,k}^{(j)} \right| < 1$$

then every sequence a_n that satisfies (8) is convergent with limit

$$\lim_{n\to\infty} a_n = \frac{T}{C+1}.$$

Proof. Set $b_n = a_n - T/(C+1)$. Then it follows by induction that for every $r \ge 0$ the sequence b_n satisfies the recurrence

$$b_{n+1} = (-1)^r \sum_{k=0}^{n-r+1} b_k P_k Q_{n,k}^{(r)} + b_0 P_0 \sum_{j=1}^{r-1} (-1)^j Q_{n,0}^{(j)} + \tilde{T}_n^{(r)}$$

for some sequence $\tilde{T}_n^{(r)}$ that tends to 0. Furthermore $Q_{n,0}^{(r)} \to 0$ as $n \to \infty$. Thus, it follows that b_n satisfies the recurrence

$$b_{n+1} = \sum_{k=0}^{n-r+1} b_k P_k \sum_{j=1}^{r} (-1)^j c_j Q_{n,k}^{(j)} + o(1),$$

too. Hence, we are in a similar situation as in Theorem 2.1, and it follows by completely the same arguments that $b_n \to 0$. ■

In Section 3 we will use this method for proving convergence for the case branching process case $P_n \sim \beta/\log n$ and $Q_n \sim 1/n$ for any $\beta \ge 1$.

3 The branching process case

3.1 Convergence

In this section we will discuss the solution a_n of (8), where $\lim_{n\to\infty} T_n = 1, P_n \sim \beta/\log n$ and $Q_n \sim 1/n$ in more detail.

Theorem 3.1 *Let $\beta > 0$ be given and suppose that the sequences P_n, Q_n, T_n are non-negative with $P_n \sim \beta/\log n$, $Q_n \sim 1/n$, and $T_n \sim 1$ as $n \to \infty$. Then every solution a_n of (8) converges to the limit*

$$\lim_{n\to\infty} a_n = \frac{1}{\beta+1}.$$

If $P_n = \beta/\log n$ (for $n \geq 2$), $Q_n = 1/(n+1)$, and $T_n = 1$ we can be even more precise.

Theorem 3.2 *Let $\beta > 0$. Then every solution of the recurrence*

$$a_{n+1} = 1 - \beta \sum_{k=2}^{n} \frac{a_k}{(n-k+1)\log k} \qquad (n \geq 2) \tag{16}$$

is convergent with limit

$$\lim_{n\to\infty} a_n = \frac{1}{\beta+1}. \tag{17}$$

Furthermore there exists an asymptotic series expansion of the form

$$a_n \sim \frac{1}{1+\beta} + \sum_{j\geq 0} c_j (\log n)^{-j} \tag{18}$$

for certain real numbers c_j (which are independent of the initial value a_2). In particular

$$a_n = \frac{1}{1+\beta} - \frac{\beta\gamma}{(1+\beta)^2 \log n} - \frac{1}{6}\frac{\beta[-6\beta\gamma^2 + \pi^2 + \beta\pi^2]}{(1+\beta)^3 \log^2 n} + O\left(\frac{1}{\log^3 n}\right)$$

We start with some easy properties.

Lemma 3.3 *Suppose that $P_n \sim \beta/\log n$ and $Q_n \sim 1/n$ as $n \to \infty$. Then for every $r \geq 1$ the sequence $Q_{n,k}^{(r)}$ satisfies*

$$Q_{n,k}^{(r)} \sim r\beta^r \frac{(\log(n-k))^{r-1}}{(n-k+1)(\log n)^{r-1}}$$

uniformly for $n/2 \leq k \leq n-r$. Furthermore we have

$$\sum_{0\leq k\leq n/2} P_k Q_{n,k}^{(r)} = O\left(\frac{1}{\log n}\right)$$

and uniformly for $\varepsilon \leq \rho \leq 1-\varepsilon$

$$\sum_{n-n^\rho \leq k \leq n-r+1} P_k Q_{n,k}^{(r)} = \beta^r \rho^r + o(1).$$

as $n \to \infty$, where $\varepsilon > 0$.

Proof. The proof is easy. We leave the detail to the reader. ∎

Lemma 3.4 *Suppose that*

$$T(x) = d_1 x - d_2 x^2 + d_3 x^3 \mp \cdots + (-1)^{r-1} d_r x^r$$

is a polynomial with (absolute) coefficients $d_j > 0$ and set

$$c_j = \frac{\frac{d_j}{\beta^j}}{\frac{d_1}{\beta} + \frac{d_2}{\beta^2} + \cdots + \frac{d_r}{\beta^r}} = -\frac{d_j}{\beta^j \cdot T\left(-\frac{1}{\beta}\right)}.$$

Then we have

$$\lim_{n\to\infty} \sum_{k=0}^{n-r+1} P_k \left| \sum_{j=1}^{r} (-1)^j c_j Q_{n,k}^{(j)} \right| = \frac{\int_0^1 |T'(x)|\, dx}{-T\left(-\frac{1}{\beta}\right)}.$$

Proof. The proof follows immediately from Lemma 3.3. ■

The final step of the proof of Theorem 3.1 is to find proper polynomials $T(x)$ with *small* integral $\int_0^1 |T'(x)|\, dx$ and *large* $|T(-1/\beta)|$. One possibility is to use *Legendre polynomials*

$$P_m(x) = \frac{1}{2^m m!} \frac{d^m}{dx^m} \left(x^2 - 1\right)^m.$$

that have all zeros in the interval $[-1, 1]$. They satisfy

$$\int_{-1}^{1} P_m(x)^2 dx = \frac{2}{2m+1} \tag{19}$$

and their generating function is given by

$$\sum_{n=0}^{\infty} P_m(x) t^m = \frac{1}{\sqrt{1 - 2xt + t^2}}. \tag{20}$$

With help of these properties we can easily prove the following estimates.

Lemma 3.5 *Let $T(x)$ be defined by $T(0) = 0$ and $T'(x) = P_m(2x-1)$, where $P_m(x)$ is the m-th Legendre polynomial. Then we have*

$$\int_0^1 |T'(x)|\, dx \le \frac{1}{\sqrt{2m+1}} \tag{21}$$

and for every (fixed) $\eta > 0$

$$|T(-\eta)| \sim \frac{\sqrt{\pi}\, (\eta(1+\eta))^{1/4}}{2m^{3/2}} (1 - 2\sqrt{\eta(1+\eta)} + 2\eta)^{-m-\frac{1}{2}} \tag{22}$$

as $m \to \infty$.

Proof. The first relation (21) follows from (19) and Cauchy's inequality.

For the proof of the second relation (22) we use the formula

$$\int_{-1}^{x} P_m(t)\, dt = -\frac{(1-x^2) P_m'(x)}{m(m+1)}$$

and obtain

$$T(x) = -\frac{2x(1-x) P_m'(2x-1)}{m(m+1)}.$$

From (20) it follows that the generating function of $P'_m(-2\eta-1)$ is given by

$$\sum_{n=0}^{\infty} P'_m(-2\eta-1)\,t^m = \frac{t}{(1+2(2\eta+1)t+t^2)^{3/2}}.$$

If $\eta > 0$ then the dominating singularity is given by

$$t_0 = 1 - 2\sqrt{\eta(1+\eta)} + 2\eta < 1$$

and the dominant behaviour around the singularity is of the form

$$\frac{1}{8t_0^{1/2}(\eta(1+\eta))^{3/4}}\left(1-\frac{t}{t_0}\right)^{-3/2}.$$

Hence we obtain

$$P'_m(-2\eta-1) \sim \frac{\sqrt{m\,\pi}}{4t_0^{1/2}(\eta(1+\eta))^{3/4}} t_0^{-m}$$

and consequently (22). ■

The proof of Theorem 3.1 is now immediate. By Lemma 3.5 for every $\beta \ge 1$ there exists m such that with $T'(x) = P_m(2x-1)$ we have

$$\frac{\int_0^1 |T'(x)|\,dx}{\left|T\left(-\frac{1}{\beta}\right)\right|} < 1.$$

(Note that $1-2\sqrt{\eta(1+\eta)}+2\eta < 1$ for all $\eta > 0$.) Hence, by combining Lemma 3.4 and Theorem 2.5 it follows that a_n is convergent.

The proof of Theorem 3.2 runs along the same lines. The only difference is that we can be a little bit more precise concering the error terms that are all of order $O(1/\log n)$. Thus it also follows that $b_n = O(1/\log n)$ and consequently $a_n = T/(1+C) + O(1/\log n)$. Finally, we can use this estimate and a simple bootstrapping procedure (via the basic recurrence (16)) to derive the asymptotic series expansion (18).

3.2 The Homogeneous Solution

In the proof of Theorem 2.4 we have used the property that every solution a_n is given by

$$a_n = a_n^{(1)} + \frac{a_2 - a_2^{(1)}}{a_2^{(2)} - a_2^{(1)}}(a_n^{(2)} - a_n^{(1)}) \quad (n \ge 2),$$

where $a_n^{(1)}$, $a_n^{(2)}$ are two different solutions. We can also say that

$$a_n = a_n^{(p)} + \frac{a_2 - a_2^{(p)}}{a_2^{(h)}} a_n^{(h)} \quad (n \ge 2),$$

where $a_n^{(p)}$ is a particular solution and $a_n^{(h)}$ is the homogeneous solution of (8), that is, the solution for $T_n = 0$:

$$a_2^{(h)} = 1, \quad a_{n+1}^{(h)} = -\beta \sum_{k=2}^{n} \frac{a_k^{(h)}}{(n-k+1)\log k} \quad (n \ge 2). \tag{23}$$

By Theorem 2.4 we can expect that there always exists a particular solution $a_n^{(p)}$ with $0 \le a_n^{(p)} \le 1$. Thus, the *shape* of the homogeneous solution $a_n^{(h)}$ describes the *structure* of all other solutions.

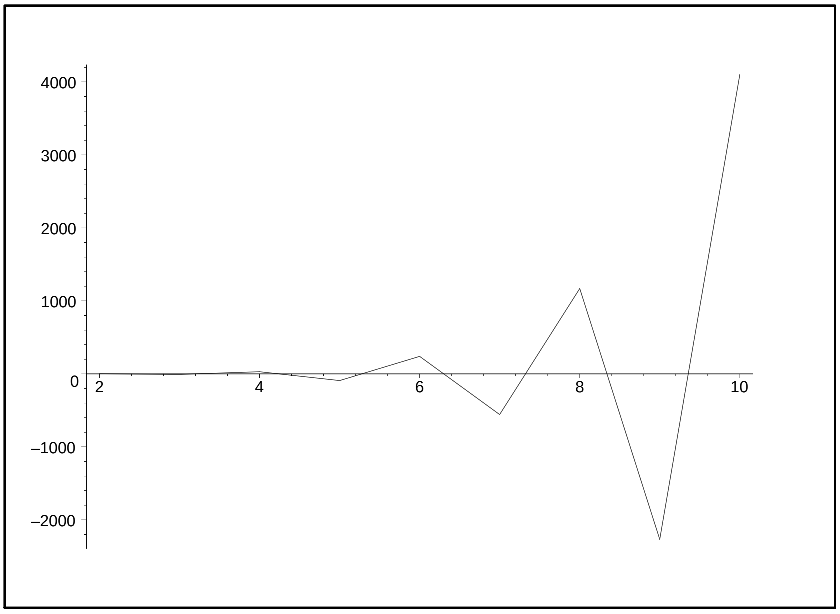

Figure 1: $a_n^{(h)} \quad (\beta = 5), n \leq 10$.

Since we know from Theorem 3.1 that every solution converges to the same limit it follows that the homogeneous solution satisfies

$$\lim_{n\to\infty} a_n^{(h)} = 0.$$

Nevertheless it is of some interest to analyze this sequence for *small* n. It turns out that it is oscillating and quite large for $n \ll 2^\beta$ (even for moderate $\beta > 1$) and gets small for $n \gg 2^\beta$.

In fact, after a few experiments, we find that we must analyse 3 regions: $R1 : n = \mathcal{O}(1)$, $R2 : n = \mathcal{O}(2^\beta)$, $R3 : n \gg \mathcal{O}(2^\beta)$.

3.2.1 Region 1: $n = \mathcal{O}(1)$

As $\beta > 1$, (23) leads to an increasing, alternating sequence, shown in Figure 1. We have choosen $\beta = 5$. $|a_n^{(h)}|$ increases first exponentially. In fact, we have for every fixed n and $\beta \to \infty$

$$a_n^{(h)} \sim (-1)^n \frac{\beta^{n-2} a_2^{(h)}}{\prod_{j=2}^{n-1} \log j}.$$

3.2.2 Region 2: $n = \mathcal{O}(2^\beta)$

We observe the alternating behaviour shown in Figure 2 (again $\beta = 5$, as in all next figures).

So we set $\tilde{a}_{2i} = a_{2i}^{(h)}, \tilde{a}_{2i+1} = -a_{2i+1}^{(h)}$. This leads to

$$\tilde{a}_{2i} = -\beta \sum_{\ell=1}^{i-1} \frac{\tilde{a}_{2\ell}}{(2i-1-2\ell+1)\log(2\ell)} + \beta \sum_{\ell=1}^{i-1} \frac{\tilde{a}_{2\ell+1}}{(2i-1-(2\ell+1)+1)\log(2\ell+1)}, \tag{24}$$

$$\tilde{a}_{2i+1} = \beta \sum_{\ell=1}^{i} \frac{\tilde{a}_{2\ell}}{(2i-2\ell+1)\log(2\ell)} - \beta \sum_{\ell=1}^{i-1} \frac{\tilde{a}_{2\ell+1}}{(2i-(2\ell+1)+1)\log(2\ell+1)}, \tag{25}$$

and $\tilde{a}_2 = 1$. The behaviour, shown in Figure 3, is now very regular.

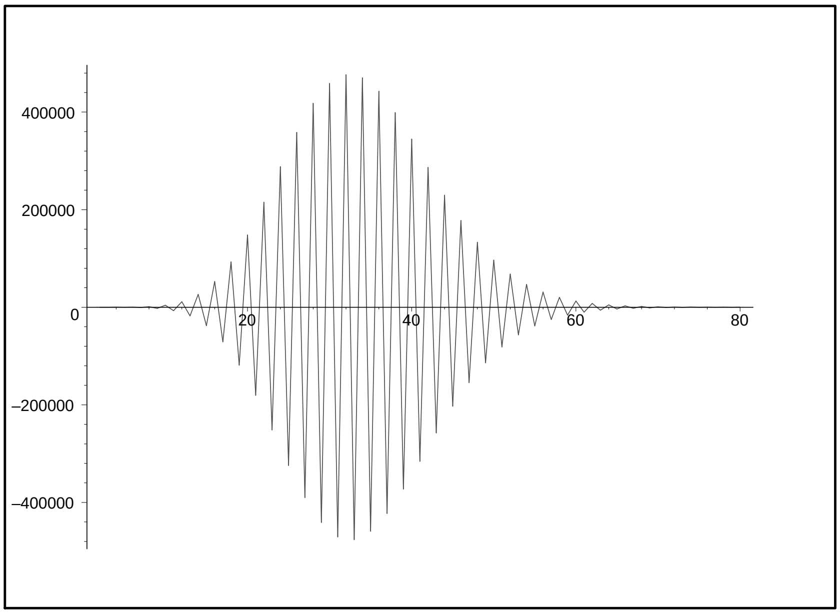

Figure 2: $a_n^{(h)} \quad (\beta = 5), n \leq 80.$

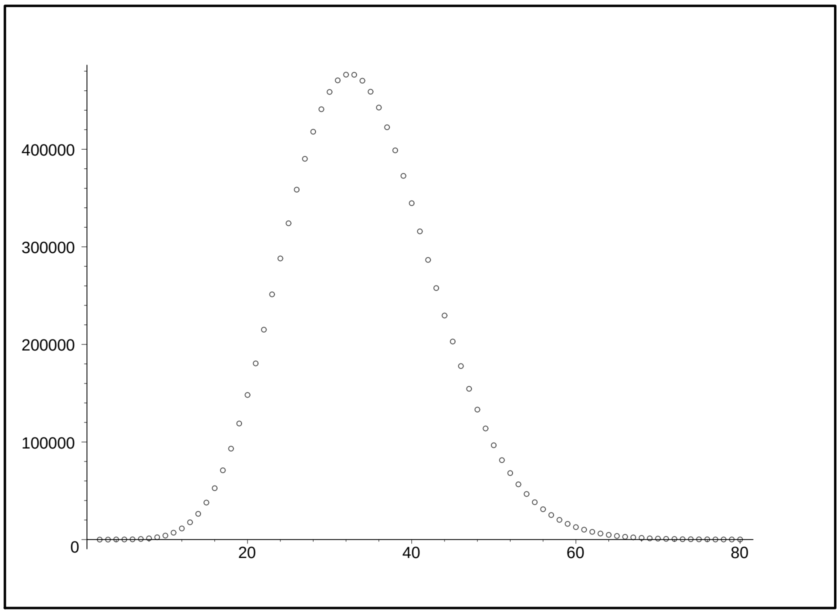

Figure 3: $\tilde{a}_n \quad (\beta = 5), n \leq 80$.

In what follows we will present some heuristics to analyze (24) and (25). We use the fact that

$$\begin{aligned}
\sum_{0}^{(n-1)/2} \frac{1}{2i+1} &= \frac{1}{2}\log n + \frac{\gamma}{2} + \frac{1}{2}\log 2 + \mathcal{O}(1/n), \\
\sum_{1}^{n/2} \frac{1}{2i} &= \frac{1}{2}\log n - \frac{1}{2}\log 2 + \frac{\gamma}{2} + \mathcal{O}(1/n), \\
\frac{1}{\log n} - \frac{1}{\log(n-i)} &= \frac{\log(1-i/n)}{\log n[\log n + \log(1-i/n)]} \sim \frac{-i}{n\log^2 n} - \frac{i^2[\log n + 2]}{n^2\log^3 n} + \mathcal{O}\left(\frac{i^3}{n^3\log^2 n}\right).
\end{aligned}$$

So we try to approximate (24) or (25) by the following first order ordinary differential equation (ODE)

$$\frac{\partial \tilde{a}(s)}{\partial s} = \left[\frac{D_1}{\log s} - 1\right]\tilde{a}(s), \tag{26}$$

with $D_1 = \beta\log 2$ and $\tilde{a}(3) = \tilde{a}_3$. The solution of (26) is given by

$$\tilde{a}_1(s) = \tilde{a}_3 \exp[-(s-3) + D_1[E_i(\log s) - E_i(\log 3)]], \tag{27}$$

and displayed in Figure 4.

The behaviour is surprisingly similar to Figure 3, taking all our approximations into account and the sensitivity to $\tilde{a}_3$. The maximum of (27) occurs at $s = 2^\beta$ (same value as for $\tilde{a}$.). A comparison is given in Figure 4.

We have observed that the two functions are more and more similar as β increases.

A better approximation can be derived as follows, using (25) and Euler-McLaurin's formula.

Set $f(i) := [\frac{\tilde{a}(s-i)}{\log(s-i)} - \frac{\tilde{a}(s)}{\log s}]$. We obtain, for even s,

$$\begin{aligned}
\tilde{a}(s+1) &\sim \beta\left[\sum_{i=2,\text{ by } 2}^{s-2} f(i) - \sum_{i=1,\text{ by } 2}^{s-3} f(i)\right] + \beta\frac{\tilde{a}(s)}{\log(s)}\left[\sum_{i=0,\text{ by } 2}^{s-2}\frac{1}{i+1} - \sum_{i=1,\text{ by } 2}^{s-3}\frac{1}{i+1}\right] \\
&\sim \beta\left\{\frac{1}{2}[f(2)-f(1)] + \frac{1}{2}[f(s-2) - f(s-3) - \frac{1}{2}\int_1^2 f(i)di\right. \\
&+ \frac{1}{2}\int_{s-3}^{s-2} f(i)di + \frac{B_2}{2}\cdot 2[f'(s-2) - f'(s-3) - f'(2) + f'(1)] \\
&+ \left.\frac{B_4}{4!}\cdot 2^3[f^{3/}(s-2) - f^{3/}(s-3) - f^{3/}(2) + f^{3/}(1)]\right\} \\
&+ \beta\frac{\tilde{a}(s)}{\log(s)}\left[\log 2 + \frac{1}{2s} + \frac{1}{4s^2}\right].
\end{aligned}$$

We now replace, in the LHS, $\tilde{a}(s+1)$ by $\tilde{a}(s) + \tilde{a}'(s) + \tilde{a}''(s)/2$ and, in the RHS, $\tilde{a}(s-i)$ by $\tilde{a}(s) - i\tilde{a}'(s) + i^2/2\tilde{a}''(s)$ in the neighbourhood of s. This leads to a 2^{nd} order ODE, which is easily (numerically) solved by MAPLE and leads to $\tilde{a}_2(s)$. Figure 4 displays $\tilde{a}_2(s)$ and $\tilde{a}$. The fit is now quite good.

3.2.3 Region 3: $n \gg 2^\beta$

Here we observe that $a_n^{(h)}$ gets small and tends to 0 (as predicted). This is shown in Figure 5.

3.2.4 Remark

Unfortunately these approximations for the homogeneous solution are not rigorous. However, they indicate that these kind of recurrences are quite interesting if one is interested in approximations that hold for all n. We have to split the positive integers into several parts and approximate $a_n^{(h)}$ by different methods. Of course this kind of approach is not new, for example compare with [5].

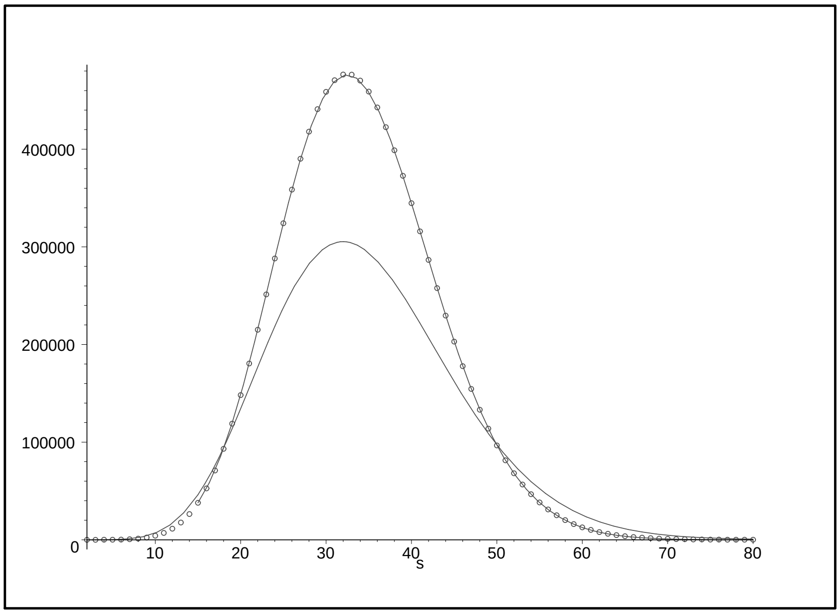

Figure 4: $\tilde{a}_1(s)$ (first order ODE, line, bottom), $\tilde{a}_2(s)$ (2^{nd} order ODE, line, top) and $\tilde{a}_{[s]}$ (circle), $(\beta = 5), n \leq 80$.

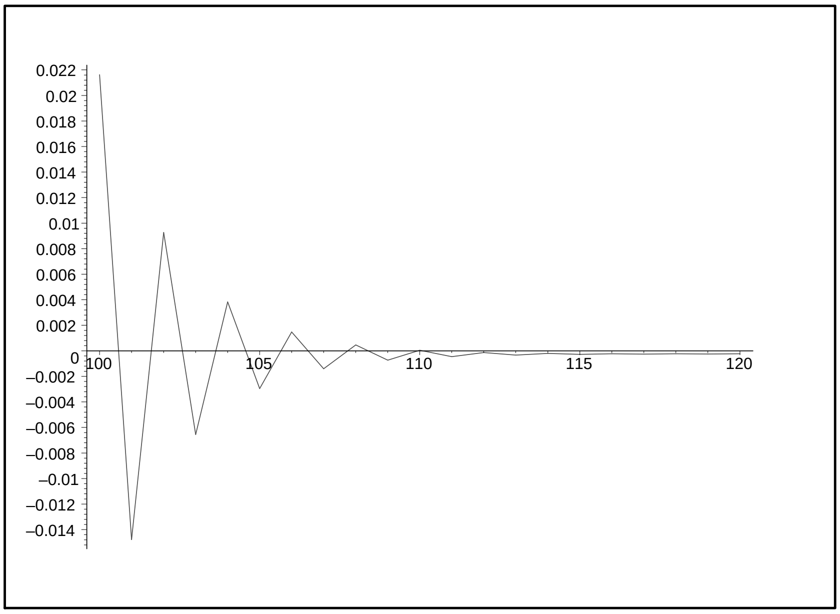

Figure 5: $a_n^{(h)}$ for $n \gg 2^\beta, \beta = 5, n < 120$

4 Conclusion

The investigation of some probabilistic structures (concerning branching stochastic processes) leads to a new analytical recurrence. This phenomenon was considered in a more general situation. Using tools from analysis, we have proved the convergence of the recurrence for all positive values of the parameter β and the asymptotic analysis of its behaviour have been done. These results have allowed us to generalize some theorems for branching processes. Finally, some heuristic approximations, based on differential equations, have also been provided.

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