On The N-Tower-Problem and Related Problems

F.Thomas Bruss* Guy Louchard † John W.Turner‡

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Abstract

N towers contain initially $n_i(i = 1, 2, ...N)$ counters. At each step a tower is chosen at random, a counter removed which is then added to another tower also chosen at random. The probability distribution for the time needed to empty one of the towers is obtained in the case $N = 3$. Arguments are set forward why no simple formulae can be expected for $N > 3$. An asymptotic expression for the mean time before one of the towers becomes empty is derived in the case of four towers when they all contain initially a comparably large number of counters. We then study related problems, in particular the ruin problem for three players. Here we use simple martingale methodology as well as a solution proposed by Ferguson for a slightly modified problem. Throughout the paper it is our main objective to shed light on the reasons why the case $N > 3$ is so substantially different from the case $N \leq 3$.

Keywords: eigenfunctions, sojourn probabilities, hitting times, harmonic analysis, martingales, ruin probabilities, Ferguson's problem, holomorphy, conformal mapping, Liouville's theorem.

1 Introduction

N towers contain initially $n_i(i = 1, 2, \dots N)$ counters. At each step a tower is chosen at random, a counter removed from this tower which is then added to another tower also chosen at random among the others. A first problem (see Engel [4] and Stirzaker [13]), is to determine the mean number of steps $T = T(n_1, n_2, \dots, n_N)$ before one of the towers becomes empty.

Exact solutions are known for the cases $N = 2$ and $N = 3$. In the former case $T = n_1 n_2$ and in the latter $T = \frac{3n_1n_2n_3}{n_1+n_2+n_3}$. The second result was first surmised (Engel [4]) from numerical simulations, then shown to obey the equation

$$\begin{aligned} T(n_1, n_2, n_3) &= 1 + \frac{1}{6}\left[T(n_1+1, n_2-1, n_3) + T(n_1-1, n_2+1, n_3)\right. \\ &+ T(n_1, n_2+1, n_3-1) + T(n_1, n_2-1, n_3+1) \\ &+ \left. T(n_1+1, n_2, n_3-1) + T(n_1-1, n_2, n_3+1)\right], \end{aligned}$$

as well as the boundary conditions ($T = 0$ if $n_1n_2n_3 = 0$) and consequently, by uniqueness, to be the exact solution.
We refer to Engel [4] for more details.

Results
The main purpose of this paper is to obtain for $N = 3$ the *complete probability distribution* for the time needed to empty one of the towers. We then derive simple forms for its mean and variance.

*Université Libre de Bruxelles, Département de Mathématique , CP 210 Boulevard du Triomphe, B-1050 Bruxelles, Belgium, email:@ulb.ac.be

†Université Libre de Bruxelles, Département d'Informatique, CP 212, Boulevard du Triomphe, B-1050 Bruxelles, Belgium, email:louchard@ulb.ac.be

‡Université Libre de Bruxelles, Département de Physique, CP 231, Boulevard du Triomphe, B-1050 Bruxelles, Belgium, email:jwturner@ulb.ac.be

Our second goal in this paper is to understand why the cases $N \leq 3$ and $N > 3$ are very different. Arguments are set forward why no simple formulae can be expected for $N > 3$. In the case $N = 4$, it will be shown that, asymptotically, when a, b, c, d are $\mathcal{O}(1)$ and M is large,

$$T(Ma, Mb, Mc, Md) \sim t_2(a,b,c,d)M^2 + t_0(a,b,c,d) + \mathcal{O}(M^{-2}),$$

where the coefficients t_2 and t_0 are expressible as series in powers of the symmetric functions $\sigma_k = a^k + b^k + c^k + d^k (k = 1,2,3,4)$. Then we shall look again shortly at $N \leq 3$ and $N > 3$ in terms of martingales. Finally we present Ferguson's solution of the ruin problem for three players which yields a different angle of view of the passage from $N \leq 3$ to $N > 3$.

2 Conditional probabilities prior to absorption $(N = 3)$

We now first concentrate on the case $N = 3$.

As the total number $C = n_1 + n_2 + n_3$ of counters is conserved, each state can be described by the two numbers $x = n_1$ and $y = n_2$, the number of counters in the third tower being $C - x - y$. As at each step the state (x, y) can change with probability $\frac{1}{6}$ into any of the states $(x \pm 1, y), (x, y \pm 1)$ or $(x \pm 1, y \mp 1)$ the problem is equivalent to the following random walk:

A particle jumps randomly at each step with probability $\frac{1}{6}$ to one of its neighbours on a regular triangular lattice. It is enclosed in a triangular domain bounded by the lines

$$x = 0, y = 0 \text{ and } x + y = C,$$

and starts at a point (a, b) (in triangular coordinates).

Let $p(x, y, n|a, b)$ be the conditional probability that the particle be at site (x, y) after n steps , knowing that initially it was at site (a, b).

Conditioning on the first transition and using the Markov property, we obtain by standard arguments

$$\begin{aligned} p(x,y,n+1|a,b) &= \frac{1}{6}\,[p(x+1,y,n|a,b) + p(x-1,y,n|a,b) + p(x,y+1,n|a,b) \\ &+ p(x,y-1,n|a,b) + p(x+1,y-1,n|a,b) + p(x-1,y+1,n|a,b)] \\ &\equiv \frac{1}{6}\Delta_{xy} p(x,y,n|a,b) \end{aligned}$$

with the absorbing boundary conditions

$$p(0,y,n|a,b) = p(x,0,n|a,b) = p(x,y,n|a,b)|_{x+y=C} = 0$$

and the initial condition

$$p(x,y,0|a,b) = \delta_{x,a}\delta_{y,b}.$$

Here, Δ_{xy} is the discrete Laplacian operator.

The idea is now to obtain eigenfunctions whose linear combination can be used to determine the probability of the random process being in a certain domain.

We first look for eigenfunctions of $\Delta_{\xi\eta}$ for ξ, η in our triangular domain.

$$\begin{aligned} \Delta_{\xi\eta} e^{i(r\xi + s\eta)} &= e^{i(r(\xi+1)+s\eta)} + e^{i(r(\xi-1)+s\eta)} + e^{i(r\xi+s(\eta+1))} \\ &+ e^{i(r\xi+s(\eta-1))} + e^{i(r(\xi+1)+s(\eta-1))} + e^{i(r(\xi-1)+s(\eta+1))} \\ &= e^{i(r\xi+s\eta)}[2\cos r + 2\cos s + 2\cos(r-s)] \end{aligned}$$

This is a generalization of the classical random walk technique as described for instance in Spitzer [12] (see section 21, propositions P1 and P2).

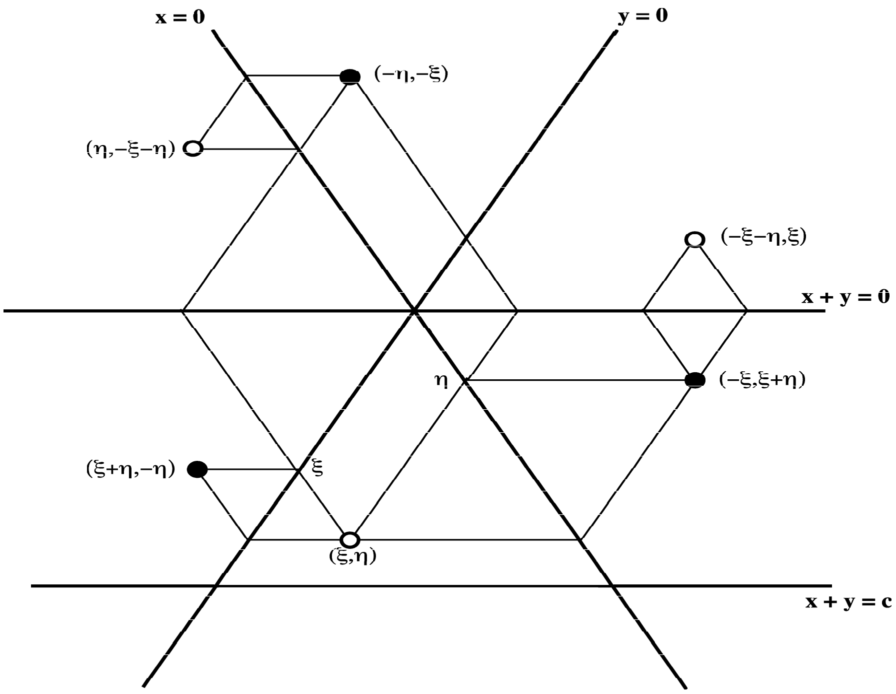

Figure 1: Triangular Domain

To obtain an eigenfunction that vanishes on the boundaries, consider the six basic images of (ξ, η) with respect to the three lines

$$x = 0, y = 0 \text{ and } x + y = 0.$$

There are (see Figure 1) three negative images $(\bullet)$ at

$$(-\xi, -\eta), (-\xi, \xi + \eta) \text{ and } (\xi + \eta, -\eta)$$

and three positive images $(\circ)$ including the original point at

$$(\xi, \eta), (-\xi - \eta, \xi) \text{ and } (\eta, -\xi - \eta).$$

The linear combination $l = l(\cdots)$ defined by

$$\begin{aligned} l &= e^{i(r\xi + s\eta)} - e^{i(-r\eta - s\xi)} + e^{i(-r(\xi+\eta)+s\xi)} - e^{i(-r\xi + s(\xi+\eta))} \\ &+ e^{i(r\eta + s(-\xi-\eta))} - e^{i(r(\xi+\eta) - s\eta)} \end{aligned}$$

is therefore an eigenfunction of $\Delta_{\xi\eta}$, corresponding to the eigenvalue

$$2\cos r + 2\cos s + 2\cos(r-s),$$

and it is readily verified that this function is identically 0 if $\xi = 0$ or $\eta = 0$, and will vanish if $\xi+\eta = C$, provided that

$$e^{iCs} = e^{-iCr} = e^{iC(r-s)}.$$

From this we obtain the conditions

$$r+s = \frac{2k\pi}{C}, r-2s = \frac{2l\pi}{C}, (k,l \text{ integers }),$$

or equivalently,

$$\begin{aligned} r &= \frac{2(2k+l)}{3C}\pi, \\ s &= \frac{2(k-l)}{3C}\pi. \end{aligned}$$

Separating real and imaginary parts of the original linear combination l leads to the following eigenfunctions for $\Delta_{\xi\eta}$:

$$\begin{aligned} u_{rs}(\xi;\eta) &= \cos(r\xi+s\eta) - \cos(r\eta+s\xi) + \cos[(r-s)\xi+r\eta] \\ &- \cos[(r-s)\xi-s\eta] + \cos[s\xi-(r-s)\eta] - \cos[r\xi+(r-s)\eta], \end{aligned}$$

and

$$\begin{aligned} g_{rs}(\xi,\eta) &= \sin(r\xi+s\eta) + \sin(r\eta+s\xi) - \sin[(r-s)\xi+r\eta] \\ &+ \sin[(r-s)\xi-s\eta] - \sin[s\xi-(r-s)\eta] - \sin[r\xi+(r-s)\eta]. \end{aligned}$$

Note that

$$\begin{aligned} u_{rs}(\xi,\eta) &= -u_{rs}(\eta,\xi) \\ g_{rs}(\xi,\eta) &= g_{rs}(\eta,\xi) \end{aligned}$$

As the domain contains $\frac{(C-1)(C-2)}{2}$ points, there can only be those number of linearly independent functions, of which $\lfloor\frac{C}{2}\rfloor\lfloor\frac{C-1}{2}\rfloor$ are symmetric with respect to the exchange of ξ and η, and of which $\lfloor\frac{C-1}{2}\rfloor\lfloor\frac{C-2}{2}\rfloor$ are antisymmetric. This can be shown to restrict the possible values for k and l to

$$\begin{aligned} k &= 1,2,\ldots,C-2 \\ l &= 1,2,\ldots,\min(k,C-1-k) \text{ for } g_{rs} \end{aligned}$$

and

$$\begin{aligned} k &= 1,2,\ldots,C-2 \\ l &= 1,2,\ldots,\min(k-1,C-1-k) \text{ for } u_{rs} \end{aligned}$$

With this restriction, the sets $\{g_{rs}\}$ and $\{u_{rs}\}$ form orthogonal bases for symmetric and antisymmetric functions in D, that is

$$\begin{aligned} \sum_{\xi,\eta\in D} g_{rs}(\xi,\eta)g_{r's'}(\xi,\eta) &= \frac{3C^2}{2}(1+\delta_{s0})\delta_{rr'}\delta_{ss'}, \\ \sum_{\xi,\eta\in D} u_{rs}(\xi,\eta)u_{r's'}(\xi,\eta) &= \frac{3C^2}{2}(1-\delta_{s0})\delta_{rr'}\delta_{ss'}, \end{aligned}$$

and furthermore

$$\sum_{\xi,\eta\in D} u_{rs}(\xi,\eta) = 0.$$

At this stage a remarkable simplification explains why the problem has a simple answer for $N=3$, that is

$$\sum_{\xi,\eta\in D} g_{rs}(\xi,\eta) = 3C\cot\frac{r}{2}\delta_{s0}. \tag{1}$$

First let $s=0$. Then $k=l$, so that $r=\frac{2k\pi}{C}$ and $e^{iCr}=1$. Therefore

$$\begin{aligned}
\sum_{\xi,\eta\in D} g_{r0}(\xi,\eta) &= 2\sum_{\xi=0}^{C}\sum_{\eta=0}^{C-\xi}[\sin(r\xi)+\sin(r\eta)-\sin(r\xi+r\eta)] \\
&= 2\Im\sum_{\xi=0}^{C}(2C+1-3\xi)e^{ir\xi} \\
&= 2\Im[(2C+1)+3C\frac{e^{ir}}{1-e^{ir}}] \\
&= 3C\cot\frac{r}{2}.
\end{aligned}$$

If $s\neq 0$, first note that

$$\sum_{\xi,\eta\in D}\sin(r\xi+s\eta) = \Im\frac{1}{1-e^{is}}\left\{\frac{1-e^{ir(C+1)}}{1-e^{ir}} - e^{is(C+1)}\frac{1-e^{i(r-s)(C+1)}}{1-e^{i(r-s)}}\right\}.$$

Substituting $r=\frac{2(2k+l)}{3C}\pi$ and $s=\frac{2(k-l)}{3C}\pi$, it turns out that in the sum for $\sum_{\xi,\eta\in D} g_{rs}$, terms corresponding to image points wrt the sides cancel two by two.

Expanding p in terms of the g_{rs} and u_{rs} functions, we obtain

$$p(x,y,n|a,b) = \sum_{r,s} A_{rs}(a,b;n)g_{rs}(x,y) + \sum_{r,s} B_{rs}(a,b;n)u_{rs}(x,y),$$

and inserting this into the equation

$$p(x,y,n+1|a,b) = \frac{1}{6}\Delta_{xy}p(x,y,n|a,b)$$

yields

$$\begin{aligned}
A_{rs}(a,b;n) &= \left[\frac{\cos r+\cos s+\cos(r-s)}{3}\right]^n A_{rs}(a,b;0) \\
B_{rs}(a,b;n) &= \left[\frac{\cos r+\cos s+\cos(r-s)}{3}\right]^n B_{rs}(a,b;0).
\end{aligned}$$

Here

$$\begin{aligned}
A_{rs}(a,b;0) &= \left[\frac{3C^2}{2}(1+\delta_{s0})\right]^{-1}\sum_{x,y\in D} g_{rs}(x,y)p(x,y,0|a,b) \\
&= \left[\frac{3C^2}{2}(1+\delta_{s0})\right]^{-1} g_{rs}(a,b),
\end{aligned}$$

and

$$\begin{aligned}
B_{rs}(a,b;0) &= \left[\frac{3C^2}{2}(1-\delta_{s0})\right]^{-1}\sum_{x,y\in D} u_{rs}(x,y)p(x,y,0|a,b) \\
&= \left[\frac{3C^2}{2}\right]^{-1} u_{rs}(a,b).
\end{aligned}$$

Therefore the probability $Q(n)$ that at time n the particle will still be inside the domain is equal to

$$\begin{aligned}
Q_{ab}(n) &= \sum_{x,y\in D} p(x,y,n|a,b) \\
&= \sum_{x,y\in D}\left[\sum_{r,s} A_{rs}(a,b;n)g_{rs}(x,y) + \sum_{r,s} B_{rs}(a,b;n)u_{rs}(x,y)\right] \\
&= \sum_{r,s} A_{rs}(a,b;n) \sum_{x,y\in D} g_{rs}(x,y) \\
&= \sum_{r,s}\left[\frac{\cos r + \cos s + \cos(r-s)}{3}\right]^n A_{rs}(a,b;0)3C\cot\frac{r}{2}\delta_{s0} \\
&= \sum_{r}\left[\frac{1+2\cos r}{3}\right]^n \frac{1}{C}\cot\frac{r}{2}g_{r0}(a,b).
\end{aligned}$$

We have thus proved the following theorem:

Theorem 2.1 *The probability $Q(n)$ that the particle be still in the domain after n steps is given by*

$$\begin{aligned}
Q(n) &= \frac{1}{C}\sum_{k=1}^{C-1}\left[\frac{1+2\cos\frac{2\pi k}{C}}{3}\right]^n \cdot \\
&\quad \cdot \frac{\sin\frac{2\pi k}{C}}{1-\cos\frac{2\pi k}{C}}\left[\sin\frac{2\pi k}{C}a + \sin\frac{2\pi k}{C}b - \sin\frac{2\pi k}{C}(a+b)\right].
\end{aligned}$$

3 Hitting time ($N = 3$)

Our next goal is to obtain the expected hitting time of the boundary, that is the mean time to absorption. It can be derived as follows.

Let $p(n)$ be the probability that absorption occurs at step n (this is equal to $Q(n-1) - Q(n)$). The generating function $q(z)$ of $Q(n)$ leads to a term $1/(1-\beta z)$, with

$$\beta := \left[\frac{1+2\cos\frac{2\pi k}{C}}{3}\right].$$

The generating function $p(z)$ of $p(n)$ is related to $q(z)$ by

$$q(z) = \frac{p(z)-1}{z-1}$$

Hence

$$q(1) = p'(1) = \mathbb{E}(T)$$

leads to a term $1/(1-\beta)$,

$$q'(1) = p''(1)/2 = \mathbb{E}[T(T-1)]/2$$

leads to a term $\beta/(1-\beta)^2$. We obtain

$$T(a,b,c) := \mathbb{E}(T) = \frac{3}{2C}\sum_{k=1}^{C-1}\frac{1}{\left[1-\cos\frac{2\pi k}{C}\right]^2}\sin\frac{2\pi k}{C}\left[\sin\frac{2\pi k}{C}a + \sin\frac{2\pi k}{C}b - \sin\frac{2\pi k}{C}(a+b)\right],$$

To compute the sum, we consider now the following integral in the complex plane

$$I = \frac{1}{2\pi i}\int_\Gamma dz \left[1-\frac{z+z^{-1}}{2}\right]^{-2} \frac{z-z^{-1}}{2i} z^{-\alpha}\frac{Cz^{C-1}}{z^C-1}$$

where Γ is a circle of radius $R>1$ surrounding the origin, and α is a positive integer. We see that

(i) there are simple poles at $z=\zeta_k=e^{\frac{2\pi i k}{C}}$ $(k=1,2,\ldots,C-1)$ where the residue of the integrand equals

$$\frac{1}{\left[1-\cos\frac{2\pi k}{C}\right]^2}\sin\frac{2\pi k}{C}e^{\frac{-2\pi i k\alpha}{C}},$$

(ii) there is a quadruple pole at $z=1$, as can be seen by rewriting the integral as

$$I = \frac{-1}{\pi}\int_\Gamma dz \frac{z+1}{(z-1)^3}\frac{Cz^{C-\alpha}}{z^C-1}.$$

The origin is a regular point (if $\alpha \le C$), and (if $a>0$) the integrand behaves as $z^{-\alpha-2}$ as $|z|\to\infty$.

Consequently

$$I=0$$

and

$$\begin{aligned}\sum_{k=1}^{C-1}\frac{1}{\left[1-\cos\frac{2\pi k}{C}\right]^2}\sin\frac{2\pi k}{C}\sin\frac{2\pi k\alpha}{C} &= -\frac{2}{3!}\frac{d^3}{dz^3}\left[(z^2-1)\frac{Cz^{C-\alpha}}{z^C-1}\right]_{z=1}\\ &= \frac{\alpha}{3}[2\alpha^2-3C\alpha+C^2].\end{aligned}$$

Finally,

$$\begin{aligned} T(a,b,c) &= \frac{2}{3C}\left[\frac{a}{3}(2a^2-3Ca+C^2)+\frac{b}{3}(2b^2-3Cb+C^2)\right.\\ &- \left.\frac{a+b}{3}(2(a+b)^2-3C(a+b)+C^2)\right]\\ &= \frac{3ab(C-a-b)}{C} = \frac{3abc}{a+b+c}.\end{aligned}$$

In a similar way the mean square time $T_2(a,b,c) := \mathbb{E}(T^2)$ can be obtained by considering the integral

$$J = \frac{1}{2\pi i}\int_\Gamma dz\left[2+\frac{z+z^{-1}}{2}\right]\left[1-\frac{z+z^{-1}}{2}\right]^{-3}\frac{z-z^{-1}}{2i}z^{-\alpha}\frac{Cz^{C-1}}{z^C-1}$$

and one finds

$$T_2(a,b,c) = \frac{3abc}{a+b+c}\frac{ab+bc+ca-1}{2}$$

Thus we have proved the following result:

Theorem 3.1 *In the 3 tower problem, with initial piles of a,b and c counters, the mean and the variance of the waiting time until one of the towers is empty are given by*

$$\begin{aligned} T_1(a,b,c) &= \frac{3abc}{a+b+c}\\ \sigma^2(a,b,c) &= \frac{3abc}{a+b+c}\left[\frac{ab+bc+ca-1}{2}-\frac{3abc}{a+b+c}\right].\end{aligned}$$

$T_1(a,b,c)$ was known (see Engel [4]). The variance does not seem to have been computed before.

4 Higher dimensions

It is stated in Engel [4] that no solution has been found for this problem when $N > 3$. It is the advantage of our approach that we now see the reason why. Consider the limiting cases when $N = 3$ and $N = 4$, that is

$$\begin{aligned} t_2(a,b,c) &:= \lim_{M\to\infty} \frac{1}{M^2} T(Ma, Mb, Mc), \\ t_2(a,b,c,d) &:= \lim_{M\to\infty} \frac{1}{M^2} T(Ma, Mb, Mc, Md). \end{aligned}$$

From the equations for the mean times $T(Ma, Mb, Mc)$ and $T(Ma, Mb, Mc, Md)$ it follows that $t_2(a,b,c)$ and $t_2(a,b,c,d)$ obey the Poisson equations

$$\left[\frac{\partial^2}{\partial a^2} + \frac{\partial^2}{\partial b^2} + \left(\frac{\partial}{\partial a} - \frac{\partial}{\partial b}\right)^2\right] t_2(a,b,c) = -6,$$

and

$$\left[\frac{\partial^2}{\partial a^2} + \frac{\partial^2}{\partial b^2} + \frac{\partial^2}{\partial c^2} + \left(\frac{\partial}{\partial a} - \frac{\partial}{\partial b}\right)^2 + \left(\frac{\partial}{\partial b} - \frac{\partial}{\partial c}\right)^2 + \left(\frac{\partial}{\partial c} - \frac{\partial}{\partial a}\right)^2\right] t_2(a,b,c,d) = -12, \tag{2}$$

with the boundary conditions

$$\begin{aligned} t_2(a,b,c) &= 0 \text{ if } a = 0 \text{ or if } b = 0 \text{ or if } a + b = C (\equiv c = 0), \\ t_2(a,b,c,d) &= 0 \text{ if } a = 0 \text{ or if } b = 0 \text{ or if } c = 0 \text{ or if } a + b + c = C (\equiv d = 0), \end{aligned}$$

respectively.

Introducing new variables through

$$\begin{aligned} a &= u + \sqrt{3}v, b = u - \sqrt{3}v, \\ a &= \sqrt{2}u + 4v, b = \sqrt{2}u - 2v - 2\sqrt{3}w, c = \sqrt{2}u - 2v + 2\sqrt{3}w, \end{aligned}$$

leads to a new set of equations (see (2))

$$\left[\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}\right] \tau(u,v) = -12,$$

and

$$\left[\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2} + \frac{\partial^2}{\partial w^2}\right] \tau(u,v,w) = -72, \tag{3}$$

with the boundary conditions

$$\tau(u,v) = 0 \text{ if } u + \sqrt{3}v = 0 \text{ or if } u - \sqrt{3}v = 0 \text{ or if } u - \frac{C}{2} = 0,$$

and

$$\begin{aligned} \tau(u,v,w) &= 0 \text{ if } \sqrt{2}u + 4v = 0 \text{ or if } \sqrt{2}u - 2v - 2\sqrt{3}w = 0 \\ &\quad \text{or if } \sqrt{2}u - 2v + 2\sqrt{3}w = 0 \text{ or if } u - \frac{C}{3\sqrt{2}} = 0. \end{aligned}$$

In the first case the domain is an equilateral triangle and the solution is well-known to be given by the lowest degree polynomial that vanishes on the edges, namely the product of their equations :

$$\tau(u,v) = -\frac{6}{C}(u^2 - 3v^2)(u - \frac{C}{2}),$$

and consequently

$$t_2(a,b,c) = \frac{3abc}{a+b+c}.$$

In the second case the domain is a tetrahedron and the simplification of the three tower case (see(1)) is lost. To see this note that

i) the product of the equations of the four faces, being of degree four, cannot be a solution of the equation, nor for that matter can any polynomial.

ii) The eigenfunctions of the Laplace operator that vanish on the boundary of a domain D can be extended analytically by reflection when the boundary contains one or several straight edges (2 dimensional case) or plane faces (3 dimensional case). If it is possible to fill space by multiple reflections and preserve parity, then the eigenfunctions are linear combinations of plane waves. This requires in particular that the vertex (or dihedral) angles be of the form π/n, n being an integer. This is true for the equilateral triangle, but not for the regular tetrahedron. See, for instance, Turner [14].

iii) the fact that the mean waiting times $T(a,b)$ and $T(a,b,c)$ are homogeneous functions of degree 2 is no longer true in the four tower case (e.g. $T(2,2,2,2) \sim 4.193$ and not 4)

In order to solve the Poisson equation (3) for $\tau(u,v,w)$ it will prove convenient to introduce a final change of variables,

$$x = \frac{12\sqrt{2}}{C}w; y = \frac{12\sqrt{2}}{C}u - 3; z = \frac{12\sqrt{2}}{C}v.$$

The equation (3) becomes then

$$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right] t_2(x,y,z) = -\frac{C^2}{4},$$

with t_2 vanishing on the surface of regular tetrahedron with vertices at

$$(0,-3,0),(0,1,2\sqrt{2})),(\sqrt{6},1,-\sqrt{2}),(-\sqrt{6},1,-\sqrt{2})).$$

The centre of gravity of the tetrahedron is now at the origin $(0,0,0)$ and its four threefold symmetry axes pass through this point.

Let $t_2(x,y,z) = -\frac{C^2(x^2+y^2+z^2)}{24} + v(x,y,z)$. As v is harmonic, it can be expanded in terms of harmonic polynomials which are invariant with respect to all elements of the complete tetrahedral group T_d(order 24).

The following procedure was adopted: first determine all harmonic polynomials of a given degree, then construct those polynomials that are invariant with respect to the tetrahedral group. Finally by the Gram-Schmidt method, construct harmonic polynomials $\{P_k\}$ which are orthogonal on the surface of the tetrahedron, i.e .

$$\int\int_{\partial T} P_m P_n d\sigma = \delta_{mn}$$

We have calculated such polynomials up to degree 10. In order to appreciate the error when expanding $v(x,y,z)$ over a finite subset of harmonic polynomials, the harmonic function $\frac{1}{r}$ was integrated over the surface of the tetrahedron t, then it was approximated as

$$\sum_{n=1}^{10} c_n P_n$$

and this sum was then integrated over t. This gives

$$\int\int_{\partial T} \frac{1}{r} d\sigma \simeq 17.7045\ldots$$

$$\int\int_{\partial T} \sum_{n-1}^{10} c_n P_n d\sigma \simeq 17.7038\ldots$$

The various changes of variables have destroyed the complete symmetry between the original ones (a, b, c, d). This however can be restored quite easily by expressing the final result for $t_2(a, b, c, d)$ in terms of the elementary symmetric functions

$$\sigma_k = a^k + b^k + c^k + d^k (k = 1, 2, 3, 4).$$

The final result reads

$$\begin{aligned}
t_2(a,b,c,d) &= \frac{\sigma_1^2 - 2\sigma_2}{4} + \frac{5}{228\sigma_1}\left(40\sigma_3 - 30\sigma_2\sigma_1 + 3\sigma_1^3\right) \\
&- \frac{35}{29013\sigma_1^2}\left(3420\sigma_4 - 4012\sigma_3\sigma_1 - 1539\sigma_2^2 + 2946\sigma_2\sigma_1^2 - 318\sigma_1^4\right) \\
&+ \frac{1}{4\sigma_1^4}\left(-3.6610292803\sigma_4\sigma_2 - 9809714552\sigma_4\sigma_1^2 + 4.4745913425\sigma_3^2\right. \\
&- 3.0508577336\sigma_3\sigma_2\sigma_1 + 2.0385912383\sigma_3\sigma_1^2 + 1.2639267753\sigma_2^3 \\
&+ \left. 1.0494295175\sigma_2^2\sigma_1^2 - 1.1791335896\sigma_2\sigma_2^2\sigma_1^4 + .1419326645\sigma_1^6\right) \\
&+ \frac{1}{4\sigma_1^5}\left(-33.5215508266\sigma_4\sigma_3 + 19.0124278104\sigma_4\sigma_2\sigma_1 - 1.18892475897\sigma_4\sigma_1^3\right. \\
&+ 41.0122273160\sigma_3^2\sigma_1 + 17.5226288412\sigma_3\sigma_2^2 - 515803385251\sigma_3\sigma_2\sigma_1^2 \\
&+ 6.1752811753\sigma_3\sigma_1^4 - 11.0260987264\sigma_2^3\sigma_1 + 18.1716326001\sigma_2^2\sigma_1^3 \\
&- \left. 4.1611751525\sigma_2\sigma_1^5 + .2577443659\sigma_1^7\right) \\
&+ \ldots
\end{aligned}$$

where all reals are given with a 10 decimal digits precision.

Further terms in the series become unfortunately too cumbersome to be given explicitly, but we have in fact calculated them up to order 10 included.

The next term t_0 of the asymptotic expansion obeys the equation

$$\Omega_1 t_0(a, b, c, d) = -\frac{1}{12}\Omega_2 t_2(a, b, c, d)$$

where

$$\begin{aligned}
\Omega_1 &:= \left[\frac{\partial^2}{\partial a^2} + \frac{\partial^2}{\partial b^2} + \frac{\partial^2}{\partial c^2} + \left(\frac{\partial}{\partial a} - \frac{\partial}{\partial b}\right)^2 + \left(\frac{\partial}{\partial b} - \frac{\partial}{\partial c}\right)^2 + \left(\frac{\partial}{\partial c} - \frac{\partial}{\partial a}\right)^2\right] \\
\Omega_2 &:= \left[\frac{\partial^4}{\partial a^4} + \frac{\partial^4}{\partial b^4} + \frac{\partial^4}{\partial c^4} + \left(\frac{\partial}{\partial a} - \frac{\partial}{\partial b}\right)^4 + \left(\frac{\partial}{\partial b} - \frac{\partial}{\partial c}\right)^4 + \left(\frac{\partial}{\partial c} - \frac{\partial}{\partial a}\right)^4\right]
\end{aligned}$$

with the same boundary conditions as for t_2. Using as before the variables (x, y, z) which centre the tetrahedron at the origin, and restricting the expansion of t_2 to its first three terms, yields the equation

$$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right] t_0(x, y, z) = \frac{245}{1018} + \text{ higher terms}$$

so that to within this approximation,

$$t_0(a, b, c, d) = -\frac{490}{509\sigma_1^2} t_2(a, b, c, d).$$

a	b	c	d	t_2	t_0	M	T_{exact}	T_{asympt}
1	1	1	1	1.0637	-.064	1	1	1.00
						2	4.193	4.19
						3	9.513	9.51
						4	16.959	16.96
						5	26.532	26.53
						6	38.233	38.23
1	1/2	1/2	1/2	.3470	-0.53	2	1.333	1.33
						4	5.499	5.50
						6	12.439	12.44
						8	22.154	22.15
1	1/3	1/3	1/3	.1680	-0.40	3	1.467	1.47
						6	6.004	6.01
						9	13.564	13.57
1	2/3	2/3	2/3	.5632	-0.60	3	5.011	5.01
						6	20.220	20.22
1	2/3	1/3	1/3	.2395	-.042	3	2.112	2.11
						6	8.581	8.58
						9	19.359	19.36
1	1	2/3	1/3	.4264	-.046	3	3.794	3.79
						6	15.308	15.30
1	1	1/2	1/2	.4794	-.051	2	1.867	1.87
						4	7.621	7.62
						6	17.209	17.21
1	3/7	3/7	2/7	.2037	-.043	7	9.934	9.94
1	1/6	1/6	1/6	.0447	-.019	6	1.577	1.59
1	1/7	1/7	1/7	.0330	-.016	7	1.587	1.60
1	1/8	1/8	1/8	.0253	-.013	8	1.594	1.61
1	1/9	1/9	1/9	.0201	-.011	9	1.598	1.62

Table 1: Exact and asymptotic mean waiting time for the case $N = 4$

The following Table 1 compares for various values of the parameters the exact value of the mean waiting time obtained by solving the recurrence (a three decimal approximation to the exact rational value is in fact given) and the approximate asymptotic value as given by $M^2 t_2 + t_0$.

The last four lines are included to show that even when the four initial piles are far from being "comparably large" the asymptotic value is still a good approximation.

5 Related Problems

We now discuss a few problems which are closely related. Our goal is to show that in those problems the difficulty in passing from dimension 3 (i.e. $N = 3$ in the tower problem) to a higher dimension has the same origin as before, that is we will always be confronted with the barriers i), ii), iii) described in Section 4.

The tower problem for $N = 2$ can of course be seen as a ruin problem for two players: in a series of independent games of two players A and B with initial capital a, respectively b, and win probability $1/2$ for each player, we suppose that the winner receives 1 unit from the loser. Here a and b are again positive integers ($a = n_1, b = n_2$), and $T = T(a, b)$ is now the expected duration of the game until A or B is ruined. This is actually the classical ruin problem for two players whose solution is well known (see e.g. Feller [5]). If W_A is the probability that A wins the sequence of games, i.e. B is ruined before

A, then

$$W_A = \frac{a}{a+b}$$

and, more generally, if A wins each game with probability p and B with the complementary probability $q = 1 - p$, then

$$W_A = \frac{\left(\frac{q}{p}\right)^{a+b} - \left(\frac{q}{p}\right)^{a}}{\left(\frac{q}{p}\right)^{a+b} - 1}$$

(see Feller, [5], Ch XIV, Section 2).

For $N = 3$ the tower problem can be considered as a ruin problem for three players in a sequence of independent games where a randomly selected player (the loser) has to pay 1 unit to a randomly selected player among the remaining two players (the winner). However the following ruin problem is more natural.

The 3 players ruin problem.
Three gamblers with initial fortunes of a, b, and c, units play a sequence of independent and fair games. In each game, the winner receives one unit from each of the other players, until one of them is ruined.

Problem 5.1 : Find the probability that a specific player is ruined.

Problem 5.2 : What is the expected number of games until at least one player is ruined.

Problem 5.1 was posed to the first-named author by T.S. Ferguson (mid-1990's). Further, Ferguson later solved, in a nice way, a close modification of this problem (see below). However, he was not really satisfied with his solution, because the solution seems to depend on a specification construction which is possible in dimension 3; it gives little insight how to generalize it for more than three players.

We first look at Problem 5.2

5.1 Solution of Problem 5.2

Let R_1 be the random time until at least one player is ruined. We want to find $\mathbb{E}(R_1)$. Remembering the article by Li [9] on the study of the occurrence of patterns, it is easy to recall little martingale tricks to find a straightforward answer to this, namely

$$\mathbb{E}(R_1) = \frac{abc}{a+b+c-2}. \tag{4}$$

Proof. Let X_n, Y_n and Z_n be the capital of player A, B, and C respectively after the nth game. Note that $X_0 = a, Y_0 = b$ and $Z_0 = c$, and that $X_n + Y_n + Z_n = a + b + c$ for all n. Also, by definition, $X_{R_1} Y_{R_1} Z_{R_1} = 0$. Since all games are fair, the winning probability equals $1/3$ for each player so that

$$\mathbb{E}(X_n Y_n Z_n | X_{n-1} = x, Y_{n-1} = y, Z_{n-1} = z) = xyz - (x + y + z) + 2 \tag{5}$$

But then, putting $K = a + b + c$,

$$\mu(n) := X_n Y_n Z_n + n(K - 2) \tag{6}$$

is a martingale. Put $\tilde{\mu}(n) = \mu(n)\mathbb{1}_{\{R_1 > n\}} + \mu(R_1)\mathbb{1}_{\{R_1 \le n\}}$. Now note that $\mathbb{E}(R_1) < \infty$ because each of the Markov chains X_n, Y_n and Z_n is bounded and has at least one absorbing state (namely 0). Since $X_n Y_n Z_n$ is bounded we have that the stopped martingale $\tilde{\mu}(n)$ is finite a.s. Therefore the optional stopping theorem (see e.g. Billingsley [2], Theorem 35.2) applies with $\mathbb{E}(\tilde{\mu}(R_1)) = \mathbb{E}(\mu(R_1)) = \mathbb{E}(\mu(0)) = X_0 Y_0 Z_0 = abc$. Hence from equation (6) and using $X_{R_1} Y_{R_1} Z_{R_1} = 0$,

$$E(R_1)(K - 2) = abc \tag{7}$$

which is equivalent to (4) since $K \geq 3$ by definition. ■

This solution (Bruss [3], unpublished) was instigated by the article of Li [9] and found independently of the work of Stirzaker [13] (whose priority is acknowledged). Stirzaker showed more however, namely

$$E(R_2) = ab + bc + ca - 2ab/(a + b + c - 2),$$

where R_2 is the time when a single player is left in the game. He also solved several other modifications in this context, in particular the expected time in a modified 4-tower problem until one tower has all the counters. (see Stirzaker [13], p.56). However, the fact that the problem is slightly modified is essential.

These methods do not seem to generalize easily to more than three players. Also, if the games are unfair, (i.e. if the win probablilities in each game are not the same for all three players) it is also not evident at all to find a suitable martingale or two or more independent martingales, even for the case $N = 3$.

5.2 Ferguson's modification for the game of three players

Ferguson later solved a modification of Problem 5.1 (1996, private communication) and we are grateful for his agreement to give his solution in the context of this paper because here the difference between the cases $N \leq 3$ and $N > 3$ appears under a different light. Ferguson modeled his problem as a Brownian motion in the plane of the equilateral triangle with barycentric coordinates (x, y, z) starting at the initial point, (a, b, c), and computed the probability that the Brownian motion first exits the triangle along one of the specified edges. The edge $z = 0$, for instance stands for the third player being ruined first. We note that the sum of the initial capitals $a + b + c = s$ remains fixed throughout the game.

The method of solution Ferguson uses here is to find a conformal mapping of the equilateral triangle onto the unit circle that maps the point (a, b, c) into the origin. Indeed, it has been known for a long time that a conformal mapping preserves the properties of a Brownian motion (see e.g. Levy [8], Theorems 56.1 and 56.2). Then the desired probability will be the proportion of the image of the edge $z = 0$ on the circumference of the circle.

We shall give a few more details and references to explain this approach. We recall that a conformal mapping preserves the relative shape of a configuration in the sense that it preserves relations of magnitude and angles in a neighbourhood of each point.

Take the equilateral triangle, Δ, in the complex plane to be the triangle with vertices $(-1, 0)$, $(1, 0)$, and $(0, i\sqrt{3})$. The mapping of Δ into the unit circle is effected in two parts; first, a map of the triangle into the upper half plane, and then a map of the upper half plane into the circle. The reason of choice of the intermediate step, that is the mapping into the half plane, is a consequence of the classical theorem of Liouville (see e.g. Theorem A.3.7 in Benedetti and Petronio [1]): although the whole plane $\mathbb{C}$ is homeomorphic to the unit disk, it is impossible to map it conformally onto the unit disk.

Take now the mapping of the triangle into the upper half plane to be the inverse map of

$$w = \frac{2}{B(1/2, 1/3)} \int_0^z \frac{dt}{(1 - t^2)^{2/3}} \tag{8}$$

where $B(\alpha, \beta)$ is the beta function, and $B(1/2, 1/3) = 2\int_0^1 (1 - t^2)^{-2/3} dt$. This maps the upper half plane into Δ, mapping $z = 0$ into $w = 0$, $z = 1$ into $w = 1$, $z = -1$ into $w = -1$, and $z = \infty$ into $w = i\sqrt{3}$. This is a special case of the Schwarz-Christoffel transformation (see e.g. Sansone and Gerretsen [11]). The conformal mapping of the upper half plane into a polygon is analyzed in Lavrentiev and Chabat, [10], p.176, Theorem 1.

After mapping w into z by the inverse of this transformation, the upper half plane is mapped into the unit disc by means of a Möbius transformation (see [11]) that maps an arbitrary point $z_0 = x_0 + iy_0$

with $y_0 > 0$ of the upper half plane into the origin. This is

$$t = \frac{z - z_0}{i(z - \overline{z}_0)}. \tag{9}$$

This maps an arbitrary point $z = x_1$ of the real axis into the point

$$t = \frac{2y_0(x_0 - x_1) + i[y_0^2 - (x_0 - x_1)^2]}{y_0^2 + (x_0 - x_1)^2} \tag{10}$$

on the unit circle. In particular, $z = x_0$ goes into $t = i$, $z = x_0 - y_0$ goes into $t = 1$, and $z = x_0 + y_0$ goes into $t = -1$. The main use of (10) is to find the arguments of the images of $z = -1$ and $z = +1$. These are

$$\begin{aligned} \theta_{-1} &= \arctan \frac{y_0^2 - (x_0 + 1)^2}{2y_0(x_0 + 1)} \\ \theta_1 &= \arctan \frac{y_0^2 - (x_0 - 1)^2}{2y_0(x_0 - 1)} \end{aligned} \tag{11}$$

respectively. The desired probability is $(\theta_1 - \theta_{-1})/2\pi$ where z_0 is the image of (a, b, c) under the inverse map (8).

The only difficulty in computation is the map (8). First consider the case of $z = iy$ pure imaginary. We find

$$\begin{aligned} w &= \frac{2}{B(1/2, 1/3)} \int_0^{iy} \frac{dt}{(1 - t^2)^{2/3}} = \frac{2i}{B(1/2, 1/3)} \int_0^{y} \frac{dx}{(1 + x^2)^{2/3}} \\ &= iB(1/2, 1/6, \frac{y^2}{1 + y^2})/B(1/2, 1/3) \end{aligned} \tag{12}$$

Here, $B(\alpha, \beta, z) = \int_0^z x^{\alpha-1}(1 - x)^{\beta-1} dx$ is the incomplete beta function. Since w must converge to the top point of Δ as y goes to ∞, we must have $B(1/2, 1/6) = \sqrt{3}B(1/2, 1/3)$. The mapping (12) becomes

$$w = i\sqrt{3}B(1/2, 1/6, \frac{y^2}{1 + y^2})/B(1/2, 1/6)$$

which is $i\sqrt{3}$ times the beta distribution function with parameters 1/2 and 1/6 evaluated at $y^2/(1+y^2)$.

As an example, Ferguson computed the probability that Player 3 is ruined when $a = b = .5$ and $c = 1$. This point in barycentric coordinates corresponds to the point $w = i\sqrt{3}/2$ in Δ. The median of the beta distribution with parameters 1/2 and 1/6 is .9510..., so we may solve $y^2/(1+y^2) = .9510\ldots$ to find $y_0 = 4.404\ldots$ and $x_0 = 0$. Substituting into (11) , one finds that $\theta_{-1} = \arctan(y_0^2 - 1)/(2y_0) = 1.1243\ldots$. Since θ_1 is placed symmetrically across the y-axis, the probability Player 3 is ruined first is $2((\pi/2) - 1.1243)/(2\pi) = .1421\ldots$.

What we should note here is again the fact that this method does not apply to more than three players because then the original triangle becomes a higher dimensional pyramid. Indeed, it is well known, that a conformal map of one open set onto another is locally given by a bi-holomorphic transformation of one complex variable, but this equivalence of bi-holomorphy and conformity breaks down for higher dimensions. (See section A3 of Benedetti and Petronio [1] for more details). Finally we note that Ferguson's approach reduced the dimension by one by describing the states of the three players as points in the triangle. Hence, here again the intrinsic difficulties start with dimension three, which echoes, as we have seen, the situation in our approaches.

5.3 Ruin as an exit on the sphere

Ferguson looked also at another version of the problem. Let (X_t, Y_t, Z_t) be Brownian motion in the upper octant of 3-space, starting at a point $(X_0, Y_0, Z_0) = (x, y, z)$ with $x > 0$, $y > 0$ and $z > 0$. The Brownian motion stops the first time the motion exits the upper octant. The problem is to find the probability that the motion exits the upper octant on the x-y plane, that is, on the plane $z = 0$. This analysis extends to any number of players.

Let T_x denote the time that a one-dimensional Brownian motion X_t first hits $x > 0$ starting at $X_0 = 0$. Using the reflection principle, (see e.g. Ito and McKean [7])

$$\begin{aligned} \Pr(T_x < t) &= 2\Pr(X_t > x) = 2\Pr(X_t\sqrt{t} > \frac{x}{\sqrt{t}}) = 2\int_{x/\sqrt{t}}^{\infty} \frac{1}{2\pi} e^{-u^2/2}\, du \\ &= \frac{1}{\sqrt{\pi}} \int_{x^2/(2t)}^{\infty} e^{-v} v^{-1/2}\, dv = 1 - I\left(\frac{1}{2}, \frac{x^2}{2t}\right) \end{aligned}$$

where $I(\alpha, x)$ represents the gamma distribution function. This yields the density

$$f_{T_x}(t) = \frac{x}{\sqrt{2\pi}} e^{-x^2/(2t)} t^{-3/2}$$

for $t > 0$. Using this, Ferguson found the probability that the upper octant is exited on the x-y plane as $\Pr(T_z < T_x, T_z < T_y)$ where T_x, T_y and T_z are independent. For more details, see Ferguson [6].

This probability is related to a gambler's ruin probability on the sphere. The projection of the Brownian motion (X_t, Y_t, Z_t) onto the sphere $x^2 + y^2 + z^2 = 1$ is a Brownian motion on the sphere. Therefore the solution of the above problem is the solution to the problem of Brownian motion on the sphere, starting at (x, y, z) on the upper octant of the sphere, of finding the probability of exiting the upper octant at $z = 0$.

In contrast to the problems we considered before, the dimension, i.e. the number of players, is here of no particular importance. The approach can be extended, in principle, to $N \geq 3$. But, clearly, this is not a real gambler's ruin problem.

6 Concluding remarks and an open question

We have obtained for $N = 3$ the *complete probability distribution* for the time needed to empty one of the towers. We then have derived simple forms for its mean and variance. Our second goal in this paper was to understand why the cases $N \leq 3$ and $N > 3$ are very different. Arguments have been set forward why no simple formulae can be expected for $N > 3$. In the case $N = 4$, we have obtain asymptotic expressions for the mean when a, b, c, d are $\mathcal{O}(M)$ and M is large. Then we have looked again shortly at $N \leq 3$ and $N > 3$ in terms of martingales. Finally we have presented Ferguson's solution of the ruin problem for three players which yields a different angle of view of the passage from $N \leq 3$ to $N > 3$.

A final word on martingales: we understand that the case $N > 3$ is analytically much more complex than the case $N \leq 3$, both in our approach and in Ferguson's approach. We understand also that this difference in complexity is a difference of nature rather than a difference of degree and that we should therefore consider the case $N \leq 3$ as a lucky coincidence which is explained in (1) for $N = 3$, and obvious for $N = 2$.

To our knowledege there is no general result linking the analytic complexity of a problem to the difficulty or impossibility of finding a suitable martingale. Hence the following question seems interesting. Will the difficulty in finding martingales to solve questions for the $N-$tower problem always echo the apparent analytic complexity or may we always hope to find independent lucky cases for martingales which will belie analytic difficulties?

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