

Conjectures about Traffic Light Queues

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ABSTRACT. In discrete time, ℓ -blocks of red lights are separated by ℓ -blocks of green lights. Cars arrive at random. The maximum line length of idle cars is fully understood for $\ell = 1$, but only partially for $2 \leq \ell \leq 3$.

Let $\ell \geq 1$ be an integer. Let $X_0 = 0$ and X_1, X_2, \dots, X_n be a sequence of independent random variables satisfying

$$\mathbb{P}\{X_i = 1\} = p, \quad \mathbb{P}\{X_i = 0\} = q \quad \text{if } i \equiv 1, 2, \dots, \ell \pmod{2\ell};$$

$$\mathbb{P}\{X_i = 0\} = p, \quad \mathbb{P}\{X_i = -1\} = q \quad \text{if } i \equiv \ell + 1, \ell + 2, \dots, 2\ell \pmod{2\ell}$$

for each $1 \leq i \leq n$. Define $S_0 = X_0$ and $S_j = \max\{S_{j-1} + X_j, 0\}$ for all $1 \leq j \leq n$. Thus cars arrive at a one-way intersection according to a Bernoulli(p) distribution; when the signal is red ($1 \leq i \leq \ell$), no cars may leave; when the signal is green ($\ell + 1 \leq i \leq 2\ell$), a car must leave (if there is one). The quantity $M_n = \max_{0 \leq j \leq n} S_j$ is the worst-case traffic congestion (as opposed to the average-case often cited). Only the circumstance when $\ell = 1$ is amenable to rigorous treatment, as far as is known. Let

$$\chi_1(p) = \frac{p(q-p)^2}{q^3}$$

where $q = 1 - p$. It is proved in [1] via a theorem in [2, 3] that

$$\mathbb{P}\{M_n \leq \log_{q^2/p^2}(n) + h\} \sim \exp\left[-\frac{\chi_1(p)}{2} \left(\frac{q^2}{p^2}\right)^{-h}\right],$$

$$\mathbb{E}(M_n) \sim \frac{\ln(n)}{\ln\left(\frac{q^2}{p^2}\right)} + \frac{\gamma + \ln\left(\frac{\chi_1(p)}{2}\right)}{\ln\left(\frac{q^2}{p^2}\right)} + \frac{1}{2} + \varphi(n), \quad \mathbb{V}(M_n) \sim \frac{\pi^2}{6} \frac{1}{\ln\left(\frac{q^2}{p^2}\right)^2} + \frac{1}{12} + \psi(n)$$

as $n \rightarrow \infty$, assuming $p < q$. The symbol γ denotes Euler's constant [4]; φ and ψ are periodic functions of $\log_{q^2/p^2}(n)$ with period 1 and of small amplitude.

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For $\ell = 2$, we have the following conjecture:

$$\mathbb{P} \{ M_n \leq \log_{q^2/p^2}(n) + h \} \sim \exp \left[-\frac{\chi_2(p)}{4} \left(\frac{q^2}{p^2} \right)^{-h} \right]$$

where

$$\chi_2(p) = \frac{(q-p)^2}{4q^6} \left[(1 - 8p^2 + 16p^3 - 8p^4) + (q-p)\sqrt{1+4pq} \right].$$

For $\ell = 3$, we likewise have:

$$\mathbb{P} \{ M_n \leq \log_{q^2/p^2}(n) + h \} \sim \exp \left[-\frac{\chi_3(p)}{6} \left(\frac{q^2}{p^2} \right)^{-h} \right]$$

where

$$\chi_3(p) = \frac{(q-p)^2}{12pq^9} \left[a + (q-p)^2 b \theta + (q-p)\sqrt{2}\sqrt{c+ab\theta} \right],$$

$$a = 1 - 4p + 10p^2 - 52p^3 + 226p^4 - 520p^5 + 640p^6 - 400p^7 + 100p^8,$$

$$b = 1 - 2p + 6p^2 - 8p^3 + 4p^4,$$

$$c = 1 - 4p + 16p^2 - 104p^3 + 506p^4 - 1808p^5 + 5604p^6 - 15576p^7 + 35574p^8 \\ - 61160p^9 + 75152p^{10} - 63440p^{11} + 34840p^{12} - 11200p^{13} + 1600p^{14},$$

$$\theta = \sqrt{1 + 4pq + 16p^2q^2}.$$

Our ad hoc technique for deriving such formulas is based on the numerical solution of a large determinantal equation, followed by the recognition of algebraic quantities given high-precision decimals. A better-justified method is currently being developed [5].

1. COMPUTATIONAL TECHNIQUE

Let $k \geq 1$ be an integer. Define $(k+1) \times (k+1)$ matrices

$$U_k = \begin{pmatrix} q & p & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & q & p & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & q & p & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & q & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & q & p & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & q & p \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & q \end{pmatrix},$$

$$V_k = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ q & p & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & q & p & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & q & p & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & p & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & q & p & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & q & p \end{pmatrix}$$

and let z_k denote the (positive real) solution of the equation

$$\det [I - U_k^\ell V_k^\ell z] = 0$$

that is closest to unity. The quantity z_k can be numerically estimated. Define

$$\chi_\ell(p) = \lim_{k \rightarrow \infty} \frac{z_k - 1}{(p/q)^{2k}}.$$

For suitably large k and sufficiently accurate z_k , the preceding ratio can be recognized as a specific algebraic number when p is rational. From a set of such algebraic numbers and corresponding p values, a $\chi_\ell(p)$ formula can be inferred.

For example, when $\ell = 3$ and $p = 1/3$, taking $k \approx 400$ and employing 100 precise digits gives

$$\chi_\ell(1/3) = \frac{1393 + 61\sqrt{217} + \sqrt{2416130 + 169946\sqrt{217}}}{6144};$$

when instead $p = 1/5$, we obtain

$$\begin{aligned} \chi_\ell(1/5) &= \frac{27 \left[18025 + 489\sqrt{1281} + 5\sqrt{25206642 + 705138\sqrt{1281}} \right]}{1048576} \\ &= \frac{9 \left[162225 + 4401\sqrt{1281} + 3\sqrt{5671494450 + 158656050\sqrt{1281}} \right]}{3145728}. \end{aligned}$$

In contrast, when $p = 1/17$, taking $k \approx 2500$ and employing 300 precise digits gives

$$\begin{aligned} \chi_\ell(1/17) &= \frac{675 \left[613160569 + 1882425\sqrt{106113} + 5\sqrt{30079190568067506 + 92338302727986\sqrt{106113}} \right]}{274877906944} \\ &= \frac{225 \left[5518445121 + 16941825\sqrt{106113} + 15\sqrt{270712715112607554 + 831044724551874\sqrt{106113}} \right]}{824633720832}; \end{aligned}$$

when instead $p = 1/19$, we obtain

$$\chi_\ell(1/19) = \frac{289 \left[13775887153 + 34281469\sqrt{161497} + 17\sqrt{1313388976733016770 + 3268219019952026\sqrt{161497}} \right]}{2380311484416}.$$

It is not hard to surmise from xy -data (with x equal to $1/p$ odd)

$$\binom{3}{6144}, \binom{5}{3145728}, \dots, \binom{17}{824633720832}, \binom{19}{2380311484416}, \dots$$

that the denominator is prescribed by

$$12(-1 + x)^9.$$

The coefficient a is found via polynomial regression on xy -data

$$\binom{3}{1393}, \binom{5}{162225}, \dots, \binom{17}{5518445121}, \binom{19}{13775887153}, \dots$$

yielding

$$y = 100 - 400x + 640x^2 - 520x^3 + 226x^4 - 52x^5 + 10x^6 - 4x^7 + x^8.$$

The coefficient b is found via data

$$\binom{3}{61}, \binom{5}{4401}, \dots, \binom{17}{16941825}, \binom{19}{34281469}, \dots$$

yielding

$$\begin{aligned} y &= 16 - 48x + 60x^2 - 40x^3 + 18x^4 - 6x^5 + x^6 \\ &= (-2 + x)^2(4 - 8x + 6x^2 - 2x^3 + x^4). \end{aligned}$$

The coefficient c is found via data

$$\binom{3}{2416130}, \binom{5}{5671494450}, \dots, \binom{17}{270712715112607554}, \binom{19}{1313388976733016770}, \dots$$

yielding

$$\begin{aligned} c &= 3200 - 22400x + 69680x^2 - 126880x^3 + 150304x^4 - 122320x^5 + 71148x^6 \\ &\quad - 31152x^7 + 11208x^8 - 3616x^9 + 1012x^{10} - 208x^{11} + 32x^{12} - 8x^{13} + 2x^{14}. \end{aligned}$$

Little insight is provided by a brute-force technique, but effectiveness is key. The greatest difficulty is knowing when cancellation of common factors has occurred and

hence needs remedy (as when $p = 1/5$ and $p = 1/17$). Semi-log plots for the coefficients were helpful in detecting such. This issue is more challenging still when $1/p$ is even, e.g., when $p = 1/10$:

$$\begin{aligned} \chi_\ell(1/10) &= \frac{64 \left[16650025 + 3818752\sqrt{19} + 80\sqrt{86608486817 + 19869473834\sqrt{19}} \right]}{1162261467} \\ &= \frac{64 \left[66600100 + 545536\sqrt{14896} + 8\sqrt{138573578907200 + 1135398504800\sqrt{14896}} \right]}{4649045868}. \end{aligned}$$

Both a factor of 28 is transferred under the radical (of 19) and a factor of 4 is introduced throughout. This similarly happens when $p = 1/12$:

$$\begin{aligned} \chi_\ell(1/12) &= \frac{100 \left[76862569 + 12636400\sqrt{37} + 20\sqrt{29539863834326 + 4856330834558\sqrt{37}} \right]}{7073843073} \\ &= \frac{100 \left[307450276 + 1805200\sqrt{29008} + 10\sqrt{1890551285396864 + 11100184764704\sqrt{29008}} \right]}{28295372292}. \end{aligned}$$

Of course, the formulas require checking for non-integer $1/p$. The case $\ell = 2$ is less tediously studied and will additionally be the focus of [5].

2. QUEUE DATA

Let $n = 10^{10}$. For each $p \in \{1/5, 1/3\}$, we generated 40000 traffic light queues (for both $\ell = 2$ and $\ell = 3$) and produced an empirical histogram for the maximum M_n . Figures 1–4 contain these histograms (in blue) along with our theoretical predictions (in red). The fit is excellent. Identical barcharts appeared in [1] but with ill-informed predictions emerging from $\ell = 1$.

3. RANDOM LIGHTS

Let Y_1, Y_2, \dots, Y_n be a sequence of independent Bernoulli(1/2) variables. Rather than defining increments X_i deterministically based on $i \bmod 2\ell$, let us define X_i randomly based on Y_i as follows:

$$\begin{aligned} \mathbb{P}\{X_i = 1\} &= p, & \mathbb{P}\{X_i = 0\} &= q & \text{if } Y_i = 1; \\ \mathbb{P}\{X_i = 0\} &= p, & \mathbb{P}\{X_i = -1\} &= q & \text{if } Y_i = 0. \end{aligned}$$

The corresponding sequence S_1, S_2, \dots, S_n , reflected at the origin as before, is a lazy random walk with expected maximum approximately equal to [6]

$$E_0(n, p) = \frac{\ln\left(\frac{n}{2}\right)}{\ln\left(\frac{q}{p}\right)} + \frac{\gamma + \ln\left(\frac{\chi_0(p)}{1}\right)}{\ln\left(\frac{q}{p}\right)} + \frac{1}{2}$$

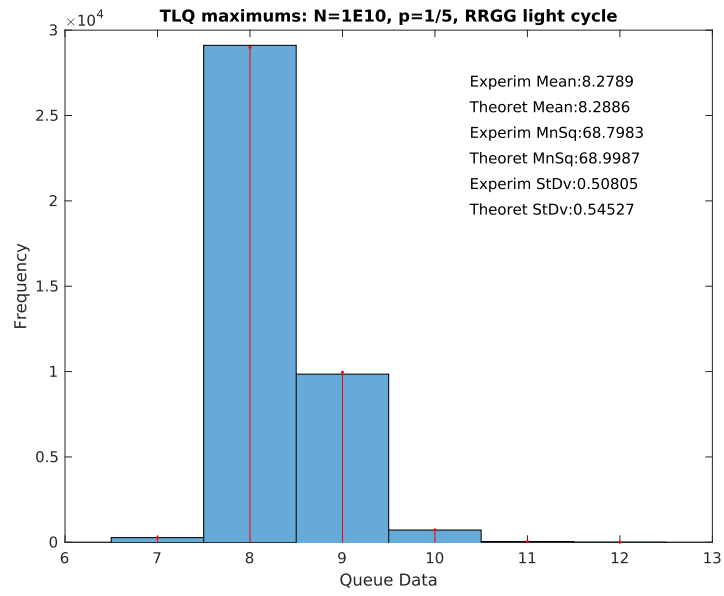


Figure 1:

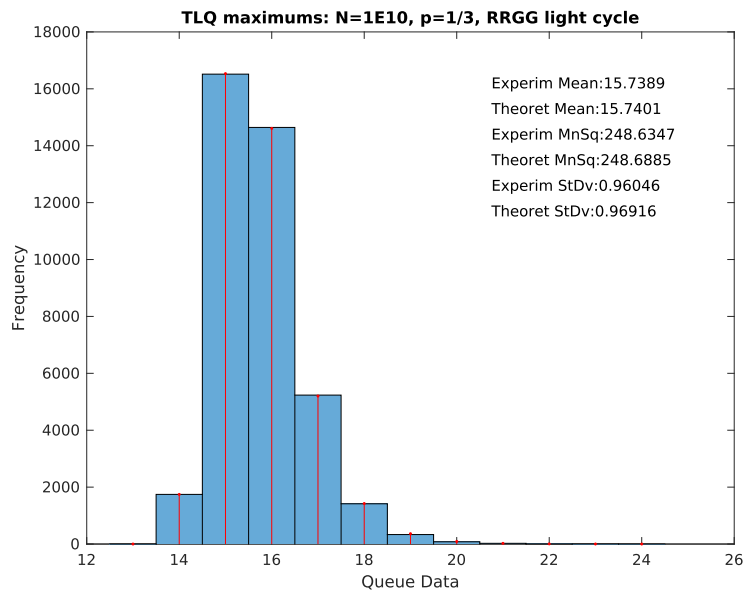


Figure 2:

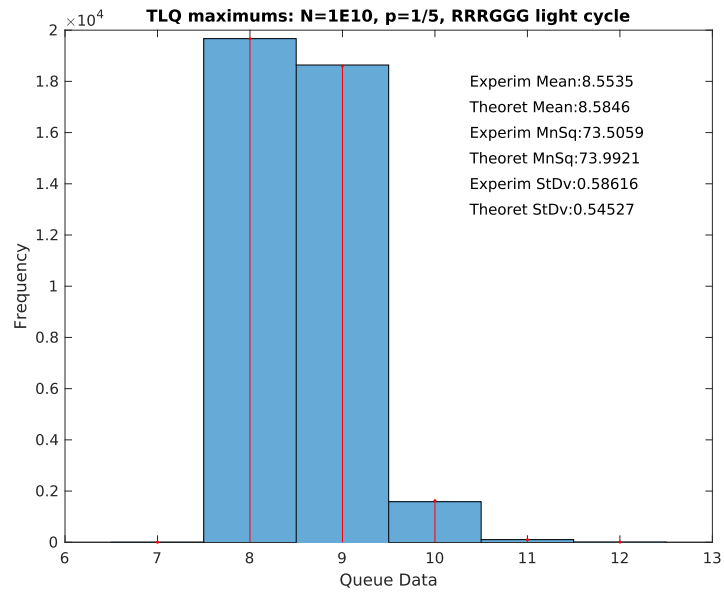


Figure 3:

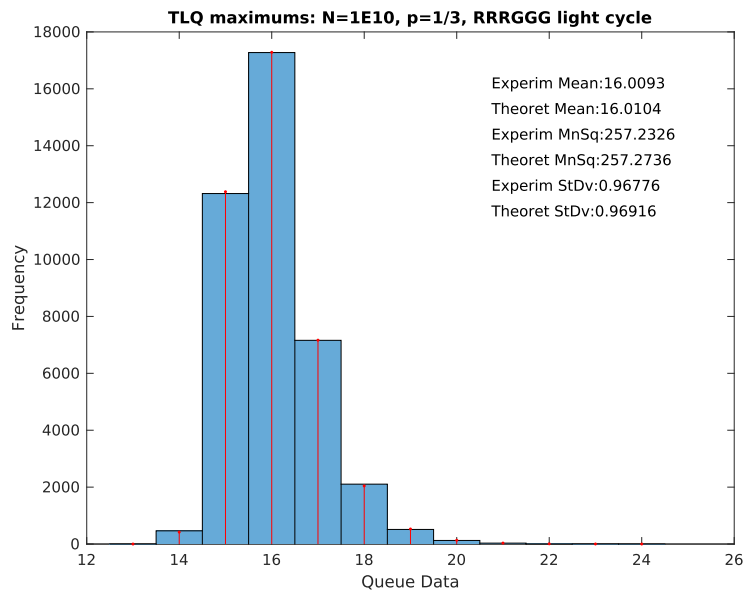


Figure 4:

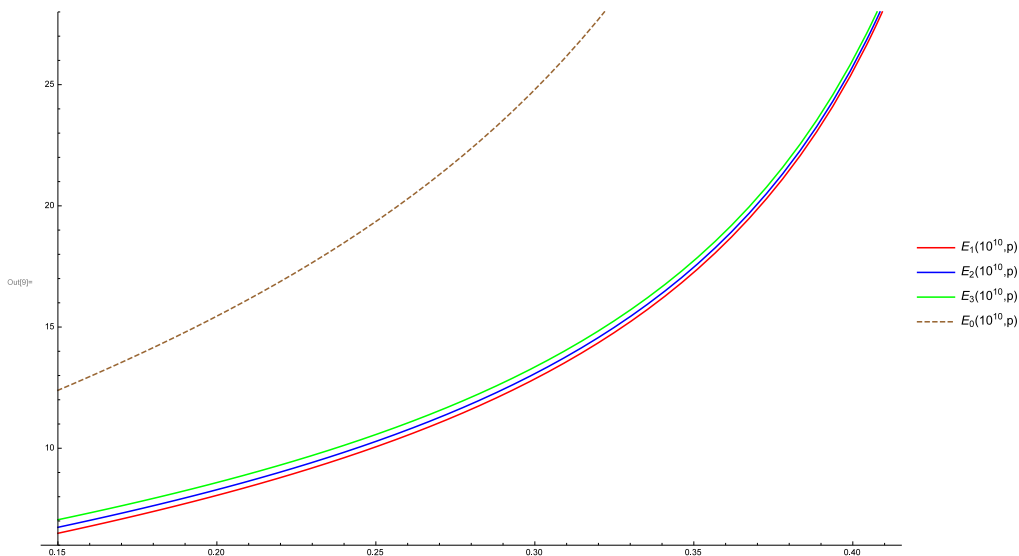


Figure 5: Expected maximums (approximate) as functions of $0.15 < p < 0.41$.

where

$$\chi_0(p) = \frac{p(q-p)^2}{q^2}.$$

Define also expected maximums associated with $\ell = 1, 2, 3$:

$$E_1(n, p) = \frac{\ln(n)}{\ln\left(\frac{q^2}{p^2}\right)} + \frac{\gamma + \ln\left(\frac{\chi_1(p)}{2}\right)}{\ln\left(\frac{q^2}{p^2}\right)} + \frac{1}{2},$$

$$E_2(n, p) = \frac{\ln(n)}{\ln\left(\frac{q^2}{p^2}\right)} + \frac{\gamma + \ln\left(\frac{\chi_2(p)}{4}\right)}{\ln\left(\frac{q^2}{p^2}\right)} + \frac{1}{2},$$

$$E_3(n, p) = \frac{\ln(n)}{\ln\left(\frac{q^2}{p^2}\right)} + \frac{\gamma + \ln\left(\frac{\chi_3(p)}{6}\right)}{\ln\left(\frac{q^2}{p^2}\right)} + \frac{1}{2}.$$

Fix $n = 10^{10}$ for sake of definiteness. Figure 5 shows that $\ell = 1$ possesses the best $E_\ell(n, p)$, in the sense of minimizing traffic backup, however only slightly compared against $\ell = 2$ and $\ell = 3$. It may be surprising that $\ell = 0$ possesses the worst $E_\ell(n, p)$ by far. Is there a red/green light strategy (either deterministic or random) that both maintains equiprobability and yet improves upon $\ell = 1$? An answer to this question would be good to see someday.

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