ADVANCING IN THE PRESENCE OF A DEMON

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Abstract. We study a parameter that contains *approximate counting,* i.e., the level reached after n random increments, driven by geometric probabilities, and *insertion costs* for *tries* as special cases. We are able to compute all moments of this parameter in a semi-automatic fashion. This is another showcase of the machinery developed in an earlier paper of these authors. Roughly speaking, it works when the underlying distributions are distributed according to the *Gumbel* distribution, or something similar.

1. Introduction

Assume that n persons want to advance on a staircase. The rules are as follows: The party starts at level 1. The m persons who advanced to level k flip a coin. Those who flip '1' (with probability q) advance to the next level; the others, who flipped '0' (with probability $p = 1 - q$) die. Additionally, there is a demon, who kills one of the survivors with probability ν, but lets them alone with probability $\mu = 1 - \nu$. The demon interfers only at a level 2 or higher. If one single person is advancing to level k and is eaten, we *don't* say that this level was reached. Only people who survive the coin flipping *and* the demon count!

As was worked out in [10], the instance $\mu = 0$ corresponds to *approximate counting.* Let us recall what it is, to keep this paper independent. There is a counter (the state the process is in at the moment), starting at 1, and random increments, the increase the counter from i to $i + 1$ with probability q^i, otherwise it stays at i. One is interested in the value of the counter after n random increments.

The other extremal case $\mu = 1$ (no demon interfering) is related to a digital data structure called *tries* [5, 9]. Although in the previous paper [10], only the symmetrical case $p = q = \frac{1}{2}$ was considered, the arguments carry over. Let p be the probability to go left (corresponding to bit 0) and q the probability to go right (corresponding to bit 1) in a trie, we think about those who go right as the *survivors,* who repeat the experiment. In this way, we always move to the right. And we are searching for an element .11111... (sufficiently many 1's), which is not present in the data structure, in other words we consider the *unsuccessful search cost, followed by an insertion* (which is the cost of inserting this element), provided that we have n random data in the trie. For the symmetric case, this makes perhaps more sense,

Key words and phrases. Gumbel distribution, approximate counting, demon, tries, search costs, moments.

H. Prodinger's research was supported by the NRF grant 2053748 and by the Center of Experimental Mathematics of the University of Stellenbosch. G. Louchard visited this center in june 2006 and thanks for its hospitality.

as we are just interested in the parameter *unsuccessful search cost,* as we are no longer considering the path that always goes right, but rather a random path.

Of course, these two special cases are not necessary to understand the paper, but they serve as a *motivation.*

The idea of introducing a probability μ of escaping the demon is borrowed from [11]; in this thesis U. Schmid studied the collision resolution schemes, related to n transmitted data, using simple tree-algorithms (Capetanakis, Hayes, Tsybakov, Mikhailov). Unlike earlier approaches, Schmid assumes that with a positive probability μ, one of the colliding packages survives and is successfully submitted; compare also [12, 13].

In the following we are interested in the random variable (RV) $K(n)$: highest level reached by (at least one member of) a party of n players. We are able to compute *all moments* (asymptotically) in an almost automatic fashion. This will be done with the techniques worked out in [7]. Note that the expectated value for the symmetric case $p = q = \frac{1}{2}$ was computed using *Rice's method* in [10].

2. Notations

We list for convenience the notations used in this paper.

$$\begin{aligned}
n &:= \text{number of persons},\\
\pi(n,k) &:= \mathbb{P}[K(n)=k], \quad \pi(n,0)=0,\ \pi(0,1)=1,\\
\Pi(n,k) &:= \sum_{i=1}^{k}\pi(n,i),\\
\nu &:= \text{Probability that the demon kills a survivor}, \quad \mu = 1-\nu,\\
q &:= \text{Probability of flipping `1' and advancing}, \quad p = 1-q,\\
F_n(u) &:= \sum_{k=1}^{\infty}\pi(n,k)u^k, \quad F_0(u)=u,
\end{aligned}$$

the generating function (GF) where the coefficient of u^k gives the probability that the party made it exactly to level k,

$$\begin{aligned}
G(z,u) &:= \sum_{n=0}^{\infty}F_n(u)\frac{z^n}{n!}, \quad G(0,u)=u,\\
D(z,u) &:= e^{-z}G(z,u) = \sum_{n=0}^{\infty}\frac{z^n}{n!}D_n(u), \quad D(0,u)=u,\\
&\quad \text{this is a classical Poissonization trick}\\
L &:= \ln 1/q,\\
\log x &:= \log_{1/q} x,\\
\widetilde{\alpha} &:= \alpha/L,\ \alpha\in\mathbb{C}
\end{aligned}$$

$$\chi_l := 2l\pi\mathbf{i}/L, \quad l \in \mathbb{Z},$$
$$\{x\} := \text{fractional part of } x.$$

Furthermore, we need a few concepts from q-analysis:

$$(x)_n := (1-x)(1-xq)\dots(1-xq^{n-1});$$

often, one writes $(x;q)_n$ to emphasize the parameter q, but that is not necessary here. $(x)_\infty := \lim_{n\to\infty}(x)_n$.

Euler's two partition identities:

$$\prod_{i=0}^{\infty}(1-tq^i) = \sum_{n=0}^{\infty}\frac{(-1)^n q^{\binom{n}{2}} t^n}{(q)_n}, \tag{1}$$
$$\prod_{i=0}^{\infty}(1-tq^i)^{-1} = \sum_{n=0}^{\infty}\frac{t^n}{(q)_n}. \tag{2}$$

They are special cases of Cauchy's formula (q-binomial theorem)

$$\frac{(at)_\infty}{(t)_\infty} = \sum_{n=0}^{\infty}\frac{(a)_n t^n}{(q)_n},$$

which we will use later. These concepts can be found in [1].

The following abbreviations will be useful:

$$Q_1 := (q)_\infty,$$
$$Q_2 := (\mu q)_\infty,$$
$$H_1(\alpha) := (e^\alpha)_\infty,$$
$$H_2(\alpha) := (\mu q e^\alpha)_\infty.$$

We use the (now standard) notation $[z^n]f(z)$ to extract the coefficient of z^n in the series expansion of $f(z)$.

3. Recurrences

Among the n persons, assume that j survive, with probability $\binom{n}{j}q^j p^{n-j}$. Among the j survivors, $j-1$ stay alive if the demon kills one of them (with probability ν) or j stay alive (with probability μ). If all of them die (with probability p^n), the highest level reached is 1.

Summing over all possible cases, we thus get the recursion

$$\pi(n,k) := \sum_{j=1}^{n}\binom{n}{j}q^j p^{n-j}[\nu\pi(j-1,k-1)+\mu\pi(j,k-1)] + p^n[\![k=1]\!].$$

The ordinary GF is given by

$$F_n(u) = u\sum_{j=1}^{n}\binom{n}{j}q^j p^{n-j}[\nu F_{j-1}(u)+\mu F_j(u)] + up^n, \quad n\ge 1, \; F_0(u)=u.$$

The exponential GF is given by

$$G\Big(\frac{z}{p},u\Big)=u\mu e^zG\Big(\frac{zq}{p},u\Big)-u^2\mu e^z+\nu\sum_{n=1}^{\infty}\frac{z^n}{n!}u\sum_{j=1}^{n}\binom{n}{j}\Big(\frac{q}{p}\Big)^jF_{j-1}(u)+ue^z.$$

Now we differentiate w.r.t. z. (The prime notation refers to this.)

$$\begin{aligned}\frac{1}{p}G'\Big(\frac{z}{p},u\Big)&=u\mu e^zG\Big(\frac{zq}{p},u\Big)+\frac{q}{p}u\mu e^zG'\Big(\frac{zq}{p},u\Big)-u^2\mu e^z\\&\quad+\nu\sum_{n=1}^{\infty}\frac{z^{n-1}}{(n-1)!}u\sum_{j=1}^{n}\binom{n}{j}\Big(\frac{q}{p}\Big)^jF_{j-1}(u)+ue^z.\end{aligned}$$

Now we *poissonize;* this translates into

$$\frac{1}{p}D'\Big(\frac{z}{p},u\Big)+\frac{q}{p}D\Big(\frac{z}{p},u\Big)=u\left[\mu\frac{q}{p}D'\Big(\frac{zq}{p},u\Big)+\frac{q}{p}D\Big(\frac{zq}{p},u\Big)\right].$$

As $D(z,u)=\sum_{n=0}^{\infty}\frac{z^n}{n!}D_n(u)$, comparing coefficients, we find

$$D_n(u)=D_{n-1}(u)(uq^n-q)/(1-u\mu q^n),$$

from which we get, upon iteration, the explicit form

$$D_n(u)=u(-1)^nq^n\frac{(u)_n}{(u\mu q)_n}.$$

Since, relating D with G,

$$F_n(u)=\sum_{j=0}^{n}\binom{n}{j}D_j(u),$$

we can continue:

$$\begin{aligned}F_n(u)&=u\sum_{j=0}^{n}\binom{n}{j}(-1)^jq^j\frac{(u)_j}{(u\mu q)_j}\\&=u\sum_{j=0}^{n}\binom{n}{j}(-1)^jq^j\frac{(u)_\infty}{(u\mu q)_\infty}\frac{(u\mu q^{j+1})_\infty}{(uq^j)_\infty}\\&=u\frac{(u)_\infty}{(u\mu q)_\infty}\sum_{j=0}^{n}\binom{n}{j}(-1)^jq^j\sum_{k=0}^{\infty}\frac{(uq^j)^k(\mu q)_k}{(q)_k}\\&=u\frac{(u)_\infty}{(u\mu q)_\infty}\sum_{k=0}^{\infty}\frac{u^k}{(q)_k}(\mu q)_k\sum_{j=0}^{n}\binom{n}{j}(-1)^jq^{j(k+1)}\\&=u\frac{(u)_\infty}{(u\mu q)_\infty}\sum_{k=0}^{\infty}\frac{u^k}{(q)_k}(\mu q)_k\big(1-q^{k+1}\big)^n.\end{aligned}$$

Reading off the coefficient $[u^l]F_n(u)$, we get the following explicit result.

Proposition 3.1.

$$\pi(n,l) = \sum_{i+j+h=l-1} \frac{(\mu q)^i}{(q)_i} \frac{(-1)^j q^{\binom{j}{2}}}{(q)_j} \frac{(\mu q)_h}{(q)_h} (1-q^{h+1})^n.$$

Note that the special case $\mu = 0$, which restricts the summation to $i = 0$, leads to

$$\sum_{j=0}^{l-1} \frac{(-1)^j q^{\binom{j}{2}}}{(q)_j (q)_{l-1-j}} (1-q^{l-j})^n$$

which is exactly Flajolet's formula [2, (46)]. We can even derive a formula with only one summation, again by invoking the q-binomial theorem:

$$\begin{aligned}
\pi(n,l) &= [u^{l-1}] \frac{(u)_\infty}{(u\mu q)_\infty} \sum_{k=0}^{\infty} \frac{u^k}{(q)_k} (\mu q)_k \big(1-q^{k+1}\big)^n \\
&= \sum_{k=0}^{l-1} \frac{(\mu q)_k}{(q)_k} \big(1-q^{k+1}\big)^n [u^{l-1-k}] \frac{(u)_\infty}{(u\mu q)_\infty} \\
&= \sum_{k=0}^{l-1} \frac{(\mu q)_k}{(q)_k} \big(1-q^{k+1}\big)^n \frac{(1/(\mu q))_{l-1-k}}{(q)_{l-1-k}} (\mu q)^{l-1-k}.
\end{aligned}$$

However, we will not use this form; one disadvantage is that for $\mu = 0$, one must consider a limit.

4. Asymptotics

Now we set $\eta = l - \log n$ and let $n \to \infty$. This gives, in the range $\eta = \mathcal{O}(1)$, the limiting distribution

$$\begin{aligned}
\pi(n,l) &\sim f(\eta) = \frac{Q_2}{Q_1} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{(\mu q)^i}{(q)_i} \frac{(-1)^j q^{\binom{j}{2}}}{(q)_j} \exp(-e^{-L\eta + L(i+j)}), \\
\Pi(n,l) &\sim H(\eta),
\end{aligned}$$

with

$$f(\eta) = H(\eta) - H(\eta - 1),$$

where we recognize the Gumbel distribution function $\exp(-e^{-x})$. To show that the limiting moments are equivalent to the moments of the limiting distribution, we need a suitable rate of convergence (in particular for large and small values of η). This is related to a uniform integrability condition (see Loève [6, Section 11.4]). For the kind of limiting distribution we consider here, the rate of convergence is analyzed in detail in [7] and [8], we will not repeat the arguments. Asymptotically, the distribution will be a periodic function of the fractional part of $\log n$. The distribution $\Pi(n,l)$ does not converge in the weak sense, it does however converge in distribution along subsequences n_m for which the fractional part of $\log n_m$ is constant.

We will use the following result from Hitczenko and Louchard [4] related to the dominant part of the moments (the '$\sim$' sign is related to the moments of the discrete RV Y_n).

Lemma 4.1. *Let a (discrete) RV Y_n be such that $\mathbb{P}(Y_n - \log n \le \eta) \sim F(\eta)$, where $F(\eta)$ is the distribution function of a continuous RV Z with mean m_1, second moment m_2. Assume that $F(\eta)$ is either an extreme-value distribution function or a convergent series of such and that we have a suitable rate of convergence. Let*

$$\varphi(\alpha) = \mathbb{E}(e^{\alpha Z}) = 1 + \sum_{k=1}^{\infty} \frac{\alpha^k}{k!} m_k.$$

Let w (with or without subscripts) denote periodic functions of $\log n$, with period 1 and with small (usually of order no more than 10^{-5}) mean and amplitude. Actually, these functions depend on the fractional part of $\log n$: $\{\log n\}$.

Then the mean of Y_n is given by

$$\begin{aligned}\mathbb{E}(Y_n - \log n) &\sim \int_{-\infty}^{+\infty} x[F(x) - F(x-1)]dx + w_1 \\ &= \widetilde{m}_1 + w_1, \quad \text{with} \quad \widetilde{m}_1 = m_1 + \tfrac{1}{2}.\end{aligned}$$

The neglected part is of order $1/n^\beta$ with $0 < \beta < 1$.

For the reader's convenience, we collect some information from [7] that we use to compute moments:

The moments of $Y_n - \log n$ are asymptotically given by $\widetilde{m}_i + w_i$, where the generating function of $\widetilde{m}_i$ is given by

$$\phi(\alpha) := \int_{-\infty}^{\infty} e^{\alpha\eta} f(\eta) d\eta = 1 + \sum_{i=1}^{\infty} \frac{\alpha^i}{i!} \widetilde{m}_i = \varphi(\alpha) \frac{e^\alpha - 1}{\alpha}. \tag{3}$$

This leads to

$$\begin{aligned}\widetilde{m}_1 &= m_1 + \frac{1}{2}, \\ \widetilde{m}_2 &= m_2 + m_1 + \frac{1}{3}, \\ \widetilde{m}_3 &= m_3 + \frac{3}{2} m_2 + m_1 + \frac{1}{4};\end{aligned}$$

To analyze the periodic component w_i to be added to the moments $\widetilde{m}_i$ we proceed as in Louchard and Prodinger [7]. For instance,

$$\mathbb{E}(Y_n - \log n) \sim E^{(1)}(n) = \sum_{j=1}^{\infty} [F(j - \log n) - F(j - \log n - 1)][j - \log n]. \tag{4}$$

Set $y = Q^{-x}$ and $G(y) = F(x)$. Equation (4) becomes

$$E^{(1)}(n) := \sum_{j=1}^{\infty} [G(n/Q^j) - G(n/Q^{j+1})][-\log(n/Q^j)],$$

the Mellin transform of which is (for a good reference on Mellin transforms, see Flajolet et al. [3] or Szpankowski [14])

$$\frac{Q^s}{1 - Q^s} \Upsilon_1^*(s), \tag{5}$$

and

$$\Upsilon_1^*(s) = \int_0^\infty y^{s-1}[G(y) - G(y/Q)][-\log y]dy$$
$$= \int_{-\infty}^\infty Q^{-sx}[F(x) - F(x-1)]xLdx.$$

Then

$$\Upsilon_1^*(s) = L\,\phi'(\alpha)\big|_{\alpha=-Ls}\,. \tag{6}$$

The fundamental strip of (5) is usually of the form $s \in \langle -C_1, 0\rangle$, $C_1 > 0$. Set also

$$\Upsilon_0^*(s) = L\,\phi(\alpha)\big|_{\alpha=-Ls}\,, \quad \Upsilon_0^*(0) = L.$$

We assume now that all poles of $\frac{Q^s}{1-Q^s}\Upsilon_1^*(s)$ are simple poles, which will be the case here, and given by $s = 0$, $s = \chi_l$, with $\chi_l := 2l\pi i/L$, $l \in \mathbb{Z} \setminus \{0\}$. Using

$$E^{(1)}(n) = \frac{1}{2\pi i}\int_{C_2-i\infty}^{C_2+i\infty} \frac{Q^s}{1-Q^s}\Upsilon_1^*(s)n^{-s}ds, \quad -C_1 < C_2 < 0,$$

the asymptotic expression of $E^{(1)}(n)$ is obtained by moving the line of integration to the right, for instance to the line $\Re s = C_4 > 0$, taking residues into account (with a negative sign). This gives

$$E^{(1)}(n) = -\mathrm{Res}\Big[\frac{Q^s}{1-Q^s}\Upsilon_1^*(s)n^{-s}\Big]\Big|_{s=0} - \sum_{l\neq 0}\mathrm{Res}\Big[\frac{Q^s}{1-Q^s}\Upsilon_1^*(s)n^{-s}\Big]\Big|_{s=\chi_l} + \mathcal{O}(n^{-C_4}).$$

The residue at $s = 0$ gives of course

$$\widetilde{m}_1 = \frac{\Upsilon_1^*(0)}{L} = \phi'(0).$$

The other residues lead to

$$w_1 = \frac{1}{L}\sum_{l\neq 0}\Upsilon_1^*(\chi_l)e^{-2l\pi i\log n}. \tag{7}$$

More generally,

$$\mathbb{E}(Y_n - \log n)^k \sim \widetilde{m}_k + w_k,$$

with

$$w_k = \frac{1}{L}\sum_{l\neq 0}\Upsilon_k^*(\chi_l)e^{-2l\pi i\log n},$$

and

$$\Upsilon_k^*(s) = L\,\phi^{(k)}(\alpha)\big|_{\alpha=-Ls}\,.$$

It will appear that $\Upsilon_k^*(s)$ are analytic functions (in some domain), depending on classical functions such as the Γ function. But we know that $\Gamma(s)$ decreases exponentially towards $\pm i\infty$:

$$|\Gamma(\sigma + it)| \sim \sqrt{2\pi}|t|^{\sigma-1/2}e^{-\pi|t|/2}, \tag{8}$$

and all our functions will also decrease exponentially towards $\pm i\infty$.

Set

$$\phi(\alpha) = \int_{-\infty}^\infty e^{\alpha\eta}f(\eta)d\eta$$

$$= \frac{Q_2}{Q_1}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(\mu q)^i}{(q)_i}\frac{(-1)^j q^{\binom{j}{2}}}{(q)_j}e^{\alpha(i+j)}\Gamma(-\widetilde{\alpha})/L = \frac{Q_2}{Q_1}\frac{H_1(\alpha)}{H_2(\alpha)}\Gamma(-\widetilde{\alpha})/L.$$

This function will be the main tool we need to derive all asymptotic moments.

5. Moments

We have

$$\mathbb{E}\left[(K(n)-\log n)^i\right] \sim \widetilde{m}_i + w_i + \mathcal{O}(n^{-\beta_i}), \quad \beta_i > 0,$$

where $\widetilde{m}_i$ are constants and w_i are periodic functions of $\log n$, with small $< 10^{-5}$ amplitude. All these expressions *only depend on $\phi(\alpha)$ and its derivatives.* For instance,

$$\begin{aligned}
\phi(0) &= 1,\\
\widetilde{m}_1 &= \phi'(0),\\
\widetilde{m}_2 &= \phi''(0),\\
w_1 &= \sum_{l\neq 0}\varphi_1(\chi_l)e^{-2l\pi\mathbf{i}\log n},\\
\varphi_1(\chi_l) &= \phi'(\alpha)|_{\alpha=-L\chi_l},\\
w_2 &= \sum_{l\neq 0}\varphi_2(\chi_l)e^{-2l\pi\mathbf{i}\log n},\\
\varphi_2(\chi_l) &= \phi''(\alpha)|_{\alpha=-L\chi_l}.
\end{aligned}$$

Also note the following local expansions for $\widetilde{\alpha}$ close to 0 resp. $-\chi_l$; recall that $\alpha = \widetilde{\alpha}L$:

$$\Gamma(-\widetilde{\alpha}) = -\frac{L}{\alpha} - \gamma - \frac{\pi^2+6\gamma^2}{12L}\alpha + \cdots,$$
$$\Gamma(-\widetilde{\alpha}) = \Gamma(\chi_l) - \frac{\psi(\chi_l)\Gamma(\chi_l)}{L}(\alpha+L\chi_l) + \frac{\Gamma(\chi_l)\big(\psi(1,\chi_l)+\psi^2(\chi_l)\big)}{2L^2}(\alpha+L\chi_l)^2 + \cdots$$

With the identities presented in the appendix identities, this leads to our main result:

Theorem 1. *The moments of the random variable $K(n)$ =* highest level reached by (at least one member of) a party of n players *satisfy the following asymptotic relation:*

$$\mathbb{E}\left[(K(n)-\log n)^i\right] \sim \widetilde{m}_i + w_i + \mathcal{O}(n^{-\beta_i}), \quad \beta_i > 0,$$
$$\begin{aligned}
\widetilde{m}_1 &= \frac{2\gamma + L - 2LC_{1,1} + 2L\mu q C_{2,1}}{L},\\
w_1 &= \sum_{l\neq 0}\varphi_1(\chi_l)e^{-2l\pi\mathbf{i}\log n},\\
\varphi_1(\chi_l) &= -\frac{\Gamma(\chi_l)}{L},
\end{aligned}$$

$$\begin{aligned}
\widetilde{m}_2 &= [\pi^2 + 6\gamma^2 + 6\gamma L - 12\gamma L C_{1,1} + 12\gamma L\mu q C_{2,1} + 2L^2 \\
&\quad - 12L^2C_{1,1} - 6L^2C_{1,2} + 6L^2C_{1,1}^2 + 12L^2\mu q C_{2,1} + 6L^2\mu^2q^2C_{2,2} \\
&\quad + 6L^2\mu^2q^2C_{2,1}^2 - 12L^2\mu q C_{2,1}C_{1,1}]/(6L^2), \\
w_2 &= \sum_{l\neq 0} \varphi_2(\chi_l) e^{-2l\pi\mathbf{i}\log n}, \\
\varphi_2(\chi_l) &= -\frac{(-2\psi(\chi_l) + L - 2LC_{1,1} + 2L\mu q C_{2,1})\Gamma(\chi_l)}{L^2}.
\end{aligned}$$

The meaning of the various constants and functions can be found in the text and the appendix.

The first two expressions are identical to Prodinger [10]. All moments can be automatically obtained by the same method.

For the reader's convenience, we explicitly write the expected value of the maximum level that a party of (initially) n people reaches:

$$\mathbb{E}(K(n)) \sim \log_{1/q}(n) + \frac{2\gamma}{L} + 1 - 2\sum_{i=1}^{\infty} \frac{q^i}{1-q^i} + 2\sum_{i=1}^{\infty} \frac{\mu q^i}{1-\mu q^i} + \delta(\log_{1/q}(n))$$

with

$$\delta(x) = -\frac{1}{L} \sum_{l\in\mathbb{Z}\setminus\{0\}} \Gamma(\chi_l) e^{-2\pi\mathbf{i}lx}.$$

6. Conclusion

This note is another showcase of the machinery developed in [7]. Once the underlying distribution is Gumbel distributed (extreme value distribution), moments can be computed in a semi-automatic way.

We hope to extend this series of applications in the near future.

Appendices.

Appendix A. Identities related to $H_1(\alpha)$

We find it useful to introduce the functions

$$\Sigma_{1,k}(z) := (k-1)! \sum_{i=1}^{\infty} q^{ki}/(1+zq^i)^k.$$

It is easily noticed that

$$\Sigma'_{1,k}(z) = -\Sigma_{1,k+1}(z).$$

The special values

$$C_{1,k} := \sum_{i=1}^{\infty} q^{ki}/(1-q^i)^k = \frac{1}{(k-1)!}\Sigma_{1,k}(-1)$$

are also of interest.

Logarithmic differentiation produces the following formulæ.

$$\begin{aligned}
(-qz)'_\infty &= (-qz)_\infty \Sigma_{1,1},\\
(-qz)''_\infty &= (-qz)_\infty [\Sigma_{1,1}^2 - \Sigma_{1,2}],\\
(-qz)'''_\infty &= (-qz)_\infty [-3\Sigma_{1,1}\Sigma_{1,2} + \Sigma_{1,1}^3 + \Sigma_{1,3}],\\
(-z)'_\infty &= (-qz)_\infty [1 + (1+z)\Sigma_{1,1}],\\
(-z)''_\infty &= (-qz)_\infty [2\Sigma_{1,1} + (1+z)[-\Sigma_{1,2} + \Sigma_{1,1}^2]],\\
(-z)'''_\infty &= (-qz)_\infty [-3\Sigma_{1,2} + 3\Sigma_{1,1}^2 + (1+z)[\Sigma_{1,3} - 3\Sigma_{1,2}\Sigma_{1,1} + \Sigma_{1,1}^3]];
\end{aligned}$$

we wrote here $\Sigma_{1,k}$ for $\Sigma_{1,k}(z)$.

Let ∂_α and ∂_z be the operators that differentiate w.r.t. α resp. z. Then we get by the chain rule for any $K(z)$, with $z = -e^\alpha$ or $z = -\mu q e^\alpha$:

$$\begin{aligned}
\partial_\alpha K &= z\partial_z K,\\
\partial_\alpha^2 K &= z[z\partial_z^2 K + \partial_z K],\\
\partial_\alpha^3 K &= z[\partial_z K + 3z\partial_z^2 K + z^2\partial_z^3 K].
\end{aligned}$$

This leads to (recall that $H_1(\alpha) = (e^\alpha)_\infty$)

$$\begin{aligned}
H_{1,0} &:= H_1(0) = 0,\\
H_{1,1} &:= \partial_\alpha H_1(\alpha)|_{\alpha=0} = -Q_1,\\
H_{1,2} &:= \partial_\alpha^2 H_1(\alpha)|_{\alpha=0} = Q_1[-1 + 2C_{1,1}],\\
H_{1,3} &:= \partial_\alpha^3 H_1(\alpha)|_{\alpha=0} = Q_1[-1 + 6C_{1,1} + 3C_{1,2} - 3C_{1,1}^2].
\end{aligned}$$

Note that we obtain the same expressions for $\alpha = -L\chi_l$, as $e^{-L\chi_l} = 1$.

Appendix B. Identities related to $H_2(\alpha)$

Now we deal with $H_2(\alpha) = (\mu q e^\alpha)_\infty$.

We need

$$\Sigma_{2,k}(z) := (k-1)! \sum_{i=0}^{\infty} q^{ki}/(1+zq^i)^k = \frac{(k-1)!}{(1+z)^k} + \Sigma_{1,k}(z)$$

and

$$C_{2,k} = \sum_{i=0}^{\infty} q^{ki}/(1-\mu q q^i)^k = \frac{1}{(k-1)!}\Sigma_{2,k}(-\mu q).$$

Since

$$\begin{aligned}
(-z)'_\infty &= (-z)_\infty \Sigma_{2,1}(z),\\
(-z)''_\infty &= (-z)_\infty [\Sigma_{2,1}^2(z) - \Sigma_{2,2}(z)],
\end{aligned}$$

we get

$$\begin{aligned}
H_{2,0} &:= H_2(0) = Q_2, \\
H_{2,1} &:= \partial_\alpha H_2(\alpha)|_{\alpha=0} = -\mu q C_{2,1} Q_2, \\
H_{2,2} &:= \partial_\alpha^2 H_2(\alpha)|_{\alpha=0} = -\mu q Q_2[C_{2,1} - \mu q(-C_{2,2} + C_{2,1}^2)].
\end{aligned}$$

Again we obtain the same expressions for $\alpha = -L\chi_l$, as $e^{-L\chi_l} = 1$.

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