A combinatorial and probabilistic study of initial- and endheights of descents in samples of geometrically distributed random variables and in permutations

Guy Louchard* Helmut Prodinger†

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Abstract

1 Introduction

Let X be a random variable (RV), distributed according to the geometric distribution with parameter p (GEOM(p)): $\mathbb{P}(X = k) = pq^{k-1}$, with $q = 1 - p$. We consider a sequence $X_1X_2 \dots X_n$ of independent RVs. We also speak about *words* $a_1 \dots a_n$; there is some interest of combinatorial parameters of such words, generated by independent geometric random variables. In this paper we continue the study of *descents*.

In a word w_1abw_2 we say that ab is the lth descent, if $a > b$ (strict model) or $a \ge b$ (weak model), and the initial word w_1 has $l-1$ descents. Furthermore, we refer to a as the *initial height* and to b as the *end height*.

In [5], these random variables where studied, but only for $l = 1$, i.e., the *first descent*. Here, we are able to deal with the general case.

This paper uses a generating functions approach and a probabilistic approach. The results complement each other, but are not disjoint. Of course, both are very useful and interesting.

The generating function approach is as follows: First, we construct a generating function $F(z, v, u)$ in 3 variables, z, v, u, where z marks the length of the word, v the number of descents, and u the last letter of the word. In other words,

$$F(z, v, u) = \sum_{n,j\ge1,\ l\ge0} \mathbb{P}[\text{a word of length } n \text{ has } l \text{ descents and last letter } j].$$

Occasionally, it is clearer to write $F(u)$ only.

*Université Libre de Bruxelles, Département d'Informatique, CP 212, Boulevard du Triomphe, B-1050 Bruxelles, Belgium. louchard@ulb.ac.be

†University of Stellenbosch, Mathematics Department, 7602 Stellenbosch, South Africa. hproding@sun.ac.za

Once this is achieved, we construct a new generating function (G for initial height, H for end height), by attaching a descent (which is a simple substitution, since the variable u "remembers" the last letter) and an arbitrary rest. In this way, we have a generating function, where the variable u no longer codes the last letter, but the initial height (resp. end height) of the lth descent.

The quantities that we get for $l = 1$ coincide (of course!) with the older paper; however, they come out in different forms. To show formally that they are the same, one uses identities from q-analysis, such as *Heine's transformation formula.*This was demonstrated extensively in [5].

We also consider the analogous questions for *ascents;* the motivation is that, in [4], the *last* descent was studied. In the reversed word, the last descent becomes the *first* ascent, and now we have developed the machinery to deal in general with the lth ascent.

The probabilistic approach works as follows. We start from an infinite sequence of Geometric random variables, with parameter p (GEOM(p)). First, we consider the successive descents as a Markov chain, related to initial and end values of each descent. Next, we use this Markov chain to obtain the distribution of inital and end values of first and second descents. Then, the first moments of the first and second descents initial values are analyzed by intensive use of some combinatorial identities. Next, we obtain the asymptotic distribution of the number of descents initial values in some interval. Finally, starting from n GEOM(p) RV, as $q \to 1$, we can derive the asymptotic properties of first and second descents, in a large permutation.

The explicit forms of the distribution of the descents becomes very complicated when going from first to second etc. descent. However, a stationary distribution, which is very simple, is rapidly approached.

In this part, only the strict model and the descents will be considered.

Several useful combinatorial identities, derived from *Heine's formula*, are given in Appendix.

We will need notation from q-analysis; the most important ones are $(x)_n := (1-x)(1-qx)\dots(1-xq^{n-1})$, and $\left[{n \atop k}\right] = \frac{(q)_n}{(q)_k(q)_{n-k}}$ (Gaussian coefficients). The relevant formulæ can be found in [1].

Combinatorial Analysis

2 Descents: the weak model

Let $f_i(u)$ be the generating function with $[z^n u^j]f_i(u)$ is the probability that a word of length $n \geq 1$ has i descents, and that the last letter is j. (Only the dependency on the variable u has been made explicit.)

Here is the recursion for $i \geq 1$.

$$f_i(u) = \frac{puz}{1-qu} f_{i-1}(1) - \frac{puz}{1-qu} f_{i-1}(uq) + \frac{puz}{1-qu} f_i(uq).$$

Now let

$$F(u) = F(z,v,u) = \sum_{i\ge0} f_i(u)v^i.$$

Then we get, by summing up,

$$F(u) - f_0(u) = \frac{puvz}{1-qu}F(1) - \frac{puvz}{1-qu}F(uq) + \frac{puz}{1-qu}F(uq) - \frac{puz}{1-qu}f_0(uq).$$

But

$$f_0(u) = \frac{puz}{1-qu} + \frac{puz}{1-qu}f_0(uq)$$

and so

$$F(u) = \frac{puz}{1-qu} + \frac{puvz}{1-qu}F(1) + \frac{puz(1-v)}{1-qu}F(uq).$$

This functional equation can be solved by iterating it:

$$F(z,v,u) = \frac{\sum_{k\ge1} \frac{(puz)^k(1-v)^k q^{\binom{k}{2}}}{(qu)_k}}{1 - v\sum_{k\ge0} \frac{(pz)^k(1-v)^k q^{\binom{k}{2}}}{(q)_k}}.$$

Now we turn to the endheights. As indicated already, we make one down step, record its height with the u-variable, and then attach anything $(1/(1-z))$. This gives the generating function

$$H(z,v,u) = \frac{puvz}{1-qu}[F(z,v,1) - F(z,v,uq)]\frac{1}{1-z} + \frac{pzu}{1-qu}.$$

The next step is to look at the behaviour for $n\to\infty$. Intuitively, it is quite clear, there should be a limit, since what happens at the lth descent must become independent of letters very far to the right. Indeed, we see that there is a simple pole at $z=1$, and the generating function

$$\Psi(v,u) = \frac{puv}{1-qu}[F(1,v,1) - F(1,v,uq)],$$

that is obtained from $H(z,v,u)$ by dropping the factor $1/(1-z)$, and the irrelevant additive term and replacing $z=1$, contains all the information. The coefficient of v^l is a probability generating function in the variable u alone. Explicit expressions become very messy, but at least the instance $l=1$ (which was studied in [5]) is manageable:

$$\begin{aligned}[v^1]\Psi(v,u) &= \frac{pu}{1-qu}[F(1,0,1) - F(1,0,uq)]\\ &= \frac{pu}{1-qu}\Bigg[\sum_{k\ge1}\frac{p^k q^{\binom{k}{2}}}{(q)_k} - \sum_{k\ge1}\frac{(pqu)^k q^{\binom{k}{2}}}{(q^2u)_k}\Bigg].\end{aligned}$$

All moments can be derived from this by differentiations. Let us just do this for the average (first moment):

$$\mathbb{E}[\text{first descent}] \sim \frac{1}{p} - \frac{\partial}{\partial u}\sum_{k\ge1}\frac{(pqu)^k q^{\binom{k}{2}}}{(q^2u)_k}\Bigg|_{u=1}$$

$$\sim \frac{1}{p} - \sum_{k\ge1} \frac{k(pq)^k q^{\binom{k}{2}}}{(q^2)_k} + \sum_{k\ge1} \frac{(pq)^k q^{\binom{k}{2}}}{(q^2)_k} \sum_{i=2}^{k+1} \frac{q^i}{1-q^i}$$

$$\sim \frac{1}{p} - \sum_{k\ge2} \frac{(k-1)p^k q^{\binom{k}{2}}}{(q)_k} - \sum_{k\ge2} \frac{p^k q^{\binom{k}{2}}}{(q)_k} \sum_{i=2}^{k} \frac{q^i}{1-q^i}.$$

Note that this average was given as

$$\frac{1}{p} - \sum_{h\ge0} (h+1)pq^{2h+1}(-p)_h$$

in [5].

For the limiting behaviour of this, as $q \to 1$, we should consider, in order to get a meaningful result, $\lim_{q\to1}(1-q)\mathbb{E}[\text{first descent}]$, which evaluates to

$$1 - \sum_{k\ge1} \frac{1}{k!} \sum_{i=2}^{k} \frac{1}{i} = 1 - \sum_{k\ge1} \frac{H_k}{k!} + \sum_{k\ge1} \frac{1}{k!} = e - \sum_{k\ge1} \frac{H_k}{k!}.$$

A similar compution can be done for *general* l, and the resulting generating function is

$$v - \frac{1}{1-v} + \frac{1}{1-ve^{1-v}} - \frac{v}{1-ve^{1-v}} \sum_{k\ge1} \frac{H_k(1-v)^{k-1}}{k!}.$$

(the coefficient of v^1 is the previous expression).

Now we turn to the *initial heights.* The same approach applies, but the substitution is even simpler. We get

$$G(z,u,v) = vz[F(z,v,u) - F(z,v,uq)]\frac{1}{1-z} + \frac{pzu}{1-qu}.$$

The limit for $n \to \infty$ leads then to the generating function

$$\Phi(v,u) = v[F(1,v,u) - F(1,v,uq)].$$

Let us compute again the instance $l = 1$; the coefficient of v^1 is particularly simple:

$$[v^1]\Phi(v,u) = F(1,0,u) - F(1,0,uq).$$

From this, we find that the average is asymptotic to

$$\frac{\partial}{\partial u}\Bigg[\sum_{k\ge1} \frac{(pu)^k q^{\binom{k}{2}}}{(qu)_k} - \sum_{k\ge1} \frac{(pqu)^k q^{\binom{k}{2}}}{(q^2u)_k}\Bigg]\Bigg|_{u=1}.$$

This checks with the expression

$$\sum_{h\ge0} (h+1)pq^{h+1}(-p)_h - \sum_{h\ge0} (2h+1)pq^{h+1}(-p)_h,$$

given in [5].

For the limit $\lim_{q\to1}(1-q)\mathbb{E}[\text{first descent}]$ we get the generating function

$$\frac{1}{v(1-ve^{1-v})}-\frac{1}{v(1-v)}-\frac{v}{1-ve^{1-v}}\sum_{k\ge1}\frac{H_k(1-v)^{k-1}}{k!}.$$

The coefficient of v in this is

$$e-1-\sum_{k\ge1}\frac{H_k}{k!}.$$

3 Descents: the strict model

Computations are similar; we only give the key steps. The functional equation is

$$F(u)=\frac{puz}{1-qu}+\frac{puvz}{1-qu}F(1)+\frac{pz(1-v)}{q(1-qu)}F(uq).$$

Thus

$$F(z,v,u)=u\sum_{k\ge1}\frac{(pz)^k(1-v)^{k-1}}{(uq)_k}\Big(1+vF(1)\Big)=\frac{u\sum_{k\ge1}\frac{(pz)^k(1-v)^k}{(uq)_k}}{1-v\sum_{k\ge0}\frac{(pz)^k(1-v)^k}{(q)_k}}.$$

Also,

$$H(z,v,u)=\left[\frac{puvz}{1-qu}F(z,v,1)-\frac{pvz}{q(1-qu)}F(z,v,uq)\right]\frac{1}{1-z}+\frac{pzu}{1-qu}.$$

Hence

$$\Psi(v,u)=\frac{puv}{1-qu}F(1,v,1)-\frac{pv}{q(1-qu)}F(1,v,uq).$$

And

$$\begin{aligned}[v^1]\Psi(v,u)&=\frac{pu}{1-qu}F(1,0,1)-\frac{p}{q(1-qu)}F(1,0,uq)\\&=\frac{pu}{1-qu}\sum_{k\ge1}\frac{p^k}{(q)_k}-\frac{pu}{1-qu}\sum_{k\ge1}\frac{p^k}{(uq^2)_k}.\end{aligned}$$

The expectation can be obtained by differentiation, followed by $u=1$:

$$\frac{1}{p}-\sum_{k\ge1}\frac{p^k}{(q)_k}\sum_{i=2}^{k}\frac{q^i}{1-q^i}.$$

The formula given in [5] is

$$\frac{1}{p}-\sum_{h\ge1}\frac{hq^{2h-1}}{(pq)_h}.$$

Now, let us compute the limit of $(1-q)\mathbb{E}$:

$$1-\sum_{k\ge1}\frac{H_k-1}{k!}=e-\sum_{k\ge1}\frac{H_k}{k!}.$$

And in general:

$$\frac{1}{1-ve^{1-v}}-\frac{1}{1-v}-\frac{v}{1-ve^{1-v}}\sum_{k\ge2}\frac{H_k(1-v)^{k-1}}{k!}.$$

Now, for the initial heights, we must consider

$$G(z,v,u)=vz[F(z,v,u)-\frac{1}{q}F(z,v,qu)]\frac{1}{1-z}+\frac{pz}{1-qu},$$

and

$$\Phi(v,u)=v[F(1,v,u)-\tfrac{1}{q}F(1,v,qu)].$$

Furthermore, to look at the first decent,

$$[v^1]\Phi(v,u)=F(1,0,u)-\tfrac{1}{q}F(1,0,qu)=u\sum_{k\ge1}\frac{p^k}{(qu)_k}-u\sum_{k\ge1}\frac{p^k}{(q^2u)_k}.$$

From this, the average, obtained by differentiation, is

$$\begin{aligned}
1+&\sum_{k\ge1}\frac{p^k}{(q)_k}\sum_{i=1}^{k}\frac{q^i}{1-q^i}-\sum_{k\ge1}\frac{p^k}{(q^2)_k}\sum_{i=2}^{k+1}\frac{q^i}{1-q^i}\\
&=1+\sum_{k\ge1}\frac{p^k}{(q)_k}\sum_{i=1}^{k}\frac{q^i}{1-q^i}-\sum_{k\ge2}\frac{p^k}{(q)_k}\sum_{i=2}^{k}\frac{q^i}{1-q^i}\\
&=1+\frac{q}{p}\sum_{k\ge1}\frac{p^k}{(q)_k}\\
&=1+\frac{q}{p}\Big(-1+\frac{1}{(p)_\infty}\Big).
\end{aligned}\tag{1}$$

The last simplification was by (A.6). The version given in [5] is

$$1+\sum_{h\ge1}\frac{hpq^{h-1}}{(pq)_h}-\sum_{h\ge1}\frac{hpq^{2h-1}}{(pq)_h}.$$

The limiting function $\lim_{q\to1}(1-q)\mathbb{E}$ is

$$\frac{v}{1-ve^{1-v}}\Bigg[\sum_{k\ge1}\frac{(1-v)^kH_k}{k!}-\sum_{k\ge1}\frac{(1-v)^k(H_{k+1}-1)}{(k+1)!}\Bigg].$$

4 Ascents: the strict model

First, we consider the case where only $a<b$ is an ascent.

Again, the treatment is very similar to before, so we only give the key steps.

Let $f_i(u)$ be the generating function with $[z^nu^j]f_i(u)$ is the probability that a word of length $n\ge1$ has i ascents, and that the last letter is j.

Here is the recursion for $i \geq 1$.

$$f_i(u) = \frac{puz}{1-qu} f_i(1) - \frac{puz}{1-qu} f_i(uq) + \frac{puz}{1-qu} f_{i-1}(uq).$$

Now let

$$F(u) = F(z,v,u) = \sum_{i\geq 0} f_i(u) v^i.$$

Then we get

$$F(u) = \frac{puz}{1-qu} + \frac{puz}{1-qu} F(1) + \frac{puz(v-1)}{1-qu} F(uq).$$

This functional equation can be solved by iterating it:

$$F(z,v,u) = \frac{\sum_{k\geq 1} \frac{(puz)^k (v-1)^{k-1} q^{\binom{k}{2}}}{(qu)_k}}{1 - \sum_{k\geq 1} \frac{(pz)^k (v-1)^{k-1} q^{\binom{k}{2}}}{(q)_k}}.$$

Further,

$$H(z,v,u) = \frac{puzv}{1-qu} F(z,v,uq) \frac{1}{1-z} + \frac{puz}{1-qu},$$

and

$$\Psi(v,u) = \frac{puv}{1-qu} F(1,v,uq).$$

The case of the first accent, i.e., the coefficient of v^1, is thus

$$[v^1]\Psi(v,u) = \frac{pu}{1-qu} F(1,0,uq) = \frac{pu}{1-qu} \frac{\sum_{k\geq 1} \frac{(pqu)^k (-1)^{k-1} q^{\binom{k}{2}}}{(q^2u)_k}}{1 - \sum_{k\geq 1} \frac{(p)^k (-1)^{k-1} q^{\binom{k}{2}}}{(q)_k}} = \frac{\sum_{k\geq 2} \frac{(pu)^k (-1)^k q^{\binom{k}{2}}}{(qu)_k}}{\sum_{k\geq 0} \frac{(p)^k (-1)^k q^{\binom{k}{2}}}{(q)_k}}.$$

Notice that

$$\sum_{k\geq 0} \frac{(p)^k (-1)^k q^{\binom{k}{2}}}{(q)_k} = (p)_\infty.$$

And therefore the average of the endheight of the first ascent tends to

$$\frac{1}{(p)_\infty} \sum_{k\geq 2} \frac{kp^k (-1)^k q^{\binom{k}{2}}}{(q)_k} + \frac{1}{(p)_\infty} \sum_{k\geq 2} \frac{p^k (-1)^k q^{\binom{k}{2}}}{(q)_k} \sum_{i=1}^{k} \frac{q^i}{1-q^i}.$$

This quantity appears in [4] in the form

$$\frac{q}{p} + \frac{1}{(p)_\infty} \sum_{h\geq 0} (h+2) p q^{2h+1} (p)_h.$$

Furthermore,

$$G(z,v,u) = vzF(z,v,uq) \frac{1}{1-z} + \frac{puz}{1-qu},$$

and

$$\Phi(v,u) = vF(1,v,uq).$$

Also,

$$[v^1]\Phi(v,u) = F(1,0,uq) = \frac{1}{(p)_\infty}\sum_{k\ge1}\frac{(pqu)^k(-1)^{k-1}q^{\binom{k}{2}}}{(q^2u)_k}.$$

And therefore the average of the initial height of the first ascent tends to

$$\frac{1}{(p)_\infty}\sum_{k\ge2}\frac{(k-1)p^k(-1)^kq^{\binom{k}{2}}}{(q)_k} + \frac{1}{(p)_\infty}\sum_{k\ge2}\frac{p^k(-1)^kq^{\binom{k}{2}}}{(q)_k}\sum_{i=2}^{k}\frac{q^i}{1-q^i}.$$

The version given in the paper [5] is

$$\frac{1}{(p)_\infty}\sum_{h\ge0}(h+1)pq^{2h+1}(p)_h.$$

5 Ascents: the weak model

$$F(u) = \frac{puz}{1-qu} + \frac{puz}{1-qu}F(1) + \frac{pz(v-1)}{q(1-qu)}F(uq),$$

$$F(z,v,u) = \frac{u\sum_{k\ge1}\frac{(pz)^k(v-1)^{k-1}}{(qu)_k}}{1-\sum_{k\ge1}\frac{(pz)^k(v-1)^{k-1}}{(q)_k}}.$$

$$H(z,v,u) = \frac{pzv}{q(1-qu)}F(z,v,uq)\frac{1}{1-z} + \frac{pz}{1-qu},$$

$$\Psi(v,u) = \frac{pv}{q(1-qu)}F(1,v,uq),$$

$$G(z,v,u) = \frac{zv}{q}F(z,v,uq)\frac{1}{1-z} + \frac{pz}{1-qu},$$

$$\Phi(v,u) = \frac{v}{q}F(1,v,uq).$$

Probabilistic Analysis

We will analyze the descents in the strict model with probabilistic tools. We start from an infinite sequence of GEOM(p) RVs.

6 Markov chains

In this section, we consider the successive descents as a Markov chain, related to initial and end values of each descent.

Let

$$\begin{aligned}
\pi(i) &:= pq^{i-1},\\
P(i) &:= \sum_{j\geq i}\pi(j) = q^{i-1},\\
I_k &:= \text{ beginning of the } k\text{th descent (initial height)},\\
J_k &:= \text{ end of the } k\text{th descent (end height)},\\
I_k &> J_k, I_{k+1}\geq J_k, I_k\geq 2.
\end{aligned}$$

By convention, J_0 is the first $\text{GEOM}(p)$ RV. We have

$$\mathbb{P}[I_2=i_2,J_2=j_2|I_1=i_1,J_1=j_1]=\sum_{l\geq 0}\sum_{j_1\leq k_2\leq\cdots\leq k_l\leq i_2}\pi(k_2)\ldots\pi(k_l)\pi(i_2)\pi(j_2),$$

with the conventions

$$\begin{aligned}
l=0 &: i_2\equiv j_1,\\
l=1 &: i_2\geq j_1.
\end{aligned}$$

This is independent of i_1.

Set

$$A(a,b,t):=\sum_{a\leq k_1\leq\cdots\leq k_t\leq b} q^{k_1+\cdots+k_t},$$

and

$$B(a,b):=\sum_{t\geq 0}\left(\frac{p}{q}\right)^t A(a,b,t).$$

Then

$$\mathbb{P}[I_2=i_2,J_2=j_2|J_1=j_1]=[\![i_2=j_1]\!]\pi(j_2)+B(j_1,i_2)\pi(i_2)\pi(j_2).$$

But we know that

$$\sum_{0\leq k_1\leq\cdots\leq k_t\leq n} q^{k_1+\cdots+k_t}=\frac{1}{(q)_t}\sum_{j=0}^{t}(-1)^j\begin{bmatrix}t\\j\end{bmatrix}q^{jn+\binom{j+1}{2}}=\begin{bmatrix}n+t\\t\end{bmatrix}.$$

Therefore

$$\begin{aligned}
A(a,b,t) &:= \sum_{a\leq k_1\leq\cdots\leq k_t\leq b} q^{k_1+\cdots+k_t}\\
&= q^{at}\sum_{0\leq k_1\leq\cdots\leq k_t\leq b-a} q^{k_1+\cdots+k_t}\\
&= q^{at}\begin{bmatrix}b-a+t\\b-a\end{bmatrix}.
\end{aligned}$$

We will use

$$\sum_{t\geq 0}\alpha^t\begin{bmatrix}m+t\\ m\end{bmatrix}=\frac{1}{(\alpha)_{m+1}}.$$

Then

$$B(a,b):=\sum_{t\geq 0}\left(\frac{p}{q}\right)^t A(a,b,t)=\frac{(p)_{a-1}}{(p)_b},$$

and

$$\mathbb{P}[I_2=i_2,J_2=j_2|J_1=j_1]=[\![i_2=j_1]\!]\pi(j_2)+\frac{(p)_{j_1-1}}{(p)_{i_2}}\pi(i_2)\pi(j_2), \tag{2}$$

$$\mathbb{P}[I_2=i_2|J_1=j_1]=[\![i_2=j_1]\!](1-q^{i_2-1})+\frac{(p)_{j_1-1}}{(p)_{i_2}}pq^{i_2-1}(1-q^{i_2-1}). \tag{3}$$

The transition matrix between I_1 and I_2 is given by

$$\begin{aligned}\mathbb{P}[I_2=i_2|I_1=i_1]&=\sum_{j_1<i_1}\frac{\pi(j_1)}{1-P(i_1)}\mathbb{P}[I_2=i_2|J_1=j_1]\\ &=[\![i_2<i_1]\!]\frac{\pi(i_2)}{1-P(i_1)}(1-q^{i_2-1})+\sum_{j_1<i_1,j_1\leq i_2}\frac{\pi(j_1)}{1-P(i_1)}\frac{(p)_{j_1-1}}{(p)_{i_2}}\pi(i_2)(1-q^{i_2-1}). \end{aligned}\tag{4}$$

Let us first check that $\sum_{i_2\geq 2}\mathbb{P}[I_2=i_2|I_1=i_1]=1$. We have, using (A.8)

$$\begin{aligned}&\sum_{i_2=2}^{i_1-1}\frac{pq^{i_2-1}}{1-q^{i_1-1}}(1-q^{i_2-1})+\sum_{i_2\geq 2}\sum_{j_1<i_1,j_1\leq i_2}\frac{pq^{j_1-1}}{1-q^{i_1-1}}\frac{(p)_{j_1-1}}{(p)_{i_2}}pq^{i_2-1}(1-q^{i_2-1})\\ &=\sum_{i_2=2}^{i_1-1}\frac{pq^{i_2-1}}{1-q^{i_1-1}}-\sum_{i_2=2}^{i_1-1}\frac{pq^{i_2-1}}{1-q^{i_1-1}}q^{i_2-1}+\sum_{i_2\geq 2}\frac{p}{1-q^{i_1-1}}\frac{1}{(p)_{i_2}}pq^{i_2-1}(1-q^{i_2-1})\\ &\quad+\sum_{j_1=2}^{i_1-1}\sum_{i_2\geq j_1}\frac{pq^{j_1-1}}{1-q^{i_1-1}}\frac{(p)_{j_1-1}}{(p)_{i_2}}pq^{i_2-1}(1-q^{i_2-1})\\ &=\frac{1}{1-q^{i_1-1}}[1-p-q^{i_1-1}]-\sum_{i_2=2}^{i_1-1}\frac{pq^{i_2-1}}{1-q^{i_1-1}}q^{i_2-1}+\frac{p}{1-q^{i_1-1}}+\frac{1}{1-q^{i_1-1}}\sum_{j_1=2}^{i_1-1}pq^{j_1-1}q^{j_1-1}=1.\end{aligned}$$

Now we compute the stationary measure $\varphi(i)$ of this matrix. (see, for instance, [3]). We have

$$\sum_{i_1>i_2}\varphi(i_1)\frac{pq^{i_2-1}}{1-q^{i_1-1}}(1-q^{i_2-1})+\sum_{i_1\geq 2}\varphi(i_1)\sum_{j_1<i_1,j_1\leq i_2}\frac{pq^{j_1-1}}{1-q^{i_1-1}}\frac{(p)_{j_1-1}}{(p)_{i_2}}pq^{i_2-1}(1-q^{i_2-1})=\varphi(i_2),$$

or, setting

$$\psi(i)=\frac{\varphi(i)}{pq^{i-1}(1-q^{i-1})},$$

we have

$$\sum_{i_1>i_2}pq^{i_1-1}\psi(i_1)+\sum_{i_1\geq 2}pq^{i_1-1}\psi(i_1)\sum_{j_1<i_1,j_1\leq i_2}pq^{j_1-1}\frac{(p)_{j_1-1}}{(p)_{i_2}}=\psi(i_2).$$

After some algebra, we will find that $\psi(i) = \mathsf{constant}$ is a solution of this equation. But it is probabilistically obvious: the stationary distribution is proportional to $pq^{i-1}(1-q^{i-1})$. We have, setting $\psi \equiv 1$,

$$\begin{aligned}
\sum_{i_1>i_2} pq^{i_1-1} + \sum_{i_1\geq 2} pq^{i_1-1} \sum_{j_1<i_1, j_1\leq i_2} pq^{j_1-1}\frac{(p)_{j_1-1}}{(p)_{i_2}} &= q^{i_2} + \frac{1}{(p)_{i_2}} \sum_{j_1=1}^{i_2} pq^{j_1-1}(p)_{j_1-1}q^{j_1} \\
&= q^{i_2} + \frac{1}{(p)_{i_2}} pq \sum_{m\geq 0} q^{2m}(p)_m - \frac{1}{(p)_{i_2}} pqq^{2i_2}(p)_{i_2} \sum_{v\geq 0} q^{2v}(pq^{i_2})_v \\
&= q^{i_2} + (pq^{i_2})_\infty + 1 - q^{i_2} - (pq^{i_2})_\infty = 1,
\end{aligned}$$

as expected. (We used (A.5) twice.)

The stationary distribution is given by

$$f_{\text{stationary}}(i) = \frac{1+q}{q} pq^{i-1}(1-q^{i-1}). \tag{5}$$

The stationary distribution generating function is given by

$$G(z) = \frac{p^2(1+q)z^2}{(1-qz)(1-q^2z)},$$

from which the stationary moments are easily derived:

$$\begin{aligned}
\mathbb{E}(I) &= \frac{2+q}{1-q^2}, \\
\mathbb{E}(I^2) &= \frac{q^3+4q+5q+4}{(1-q^2)^2}.
\end{aligned}$$

The stationay distribution of the end value is given by

$$(1+q)/q \sum_{i_2+1}^{\infty} pq^{i_1-1}(1-q^{i_1-1})pq^{i_2-1}/(1-q^{i_1-1}) = (1+q)pq^{2i_2-2}, i_2 = 1\ldots\infty,$$

which sums correctly to 1.

Another transition matrix of interest is given by

$$\mathbb{P}[J_2 = j_2 | J_1 = j_1] = \pi(j_2)[\![j_2 < j_1]\!] + \sum_{i_2\geq j_1, i_2>j_2} \frac{(p)_{j_1-1}}{(p)_{i_2}} \pi(i_2)\pi(j_2). \tag{6}$$

7 Descent distributions

In this section, we use the Markov chain derived in the previous section to obtain the distribution of inital and end values of first and second descents. For the first descent initial value I_1, for example, with (A.9), we have

$$f_1(i_1) := \mathbb{P}[I_1 = i_1] = \sum_{j_0=1}^{i_1} pq^{j_0-1}\mathbb{P}[I_1 = i_1 | J_0 = j_0]$$

$$= pq^{i_1-1}(1-q^{i_1-1}) + \sum_{j_0=1}^{i_1} pq^{j_0-1}\frac{(p)_{j_0-1}}{(p)_{i_1}}pq^{i_1-1}(1-q^{i_1-1})$$
$$= pq^{i_1-1}(1-q^{i_1-1})\frac{1}{(p)_{i_1}}. \tag{7}$$

A graph of $f_1(i)$, $q = 0.7$ is given in Figure 1. It is easily checked that

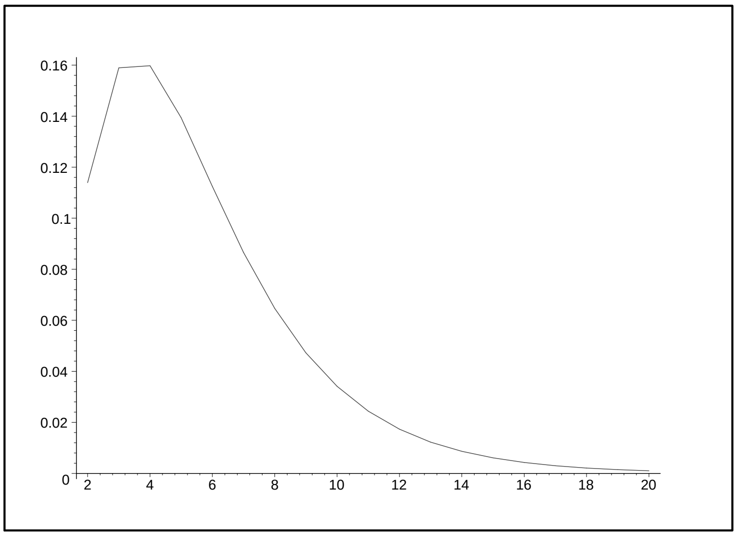


Figure 1: $f_1(i), q = 0.7$

$$\sum_{i=2}^{\infty} f_1(i) = 1.$$

Indeed
$$\sum_{i\geq 2} pq^{i-1}(1-q^{i-1})\frac{1}{(p)_i} = 1, \text{ by (A.4).}$$

A comparison between $f_1(i)$ and the stationary distribution $f_{\text{stationary}}(i)$, $q = 0.7$ is given in Figure 2.

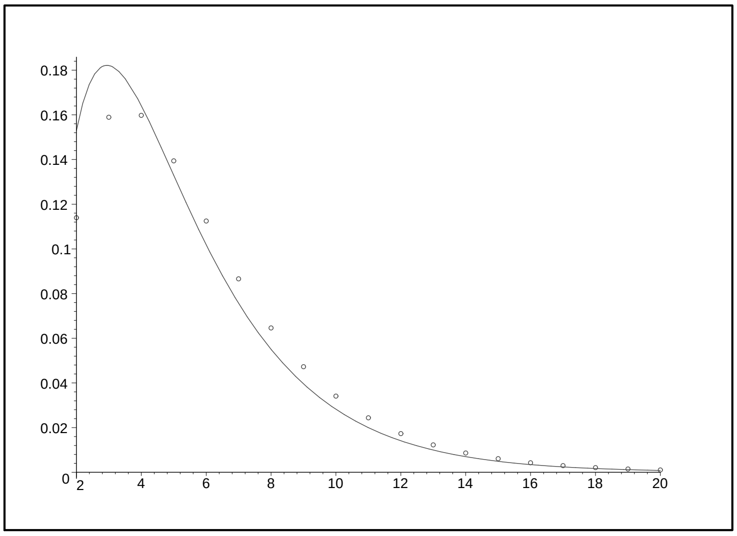
For the end height of the first descent, we have, with (A.9) and (A.10),

$$\gamma_1(j_1) := \mathbb{P}[J_1 = j_1] = \sum_{j_0>j_1} pq^{j_0-1}pq^{j_1-1} + \sum_{i_1>j_1}\sum_{j_0=1}^{i_1} pq^{j_0-1}\frac{(p)_{j_0-1}}{(p)_{i_1}}pq^{i_1-1}pq^{j_1-1}$$
$$= pq^{j_1-1}\left[\frac{1}{(p)_\infty} - \frac{1}{(p)_{j_1}}\right]. \tag{8}$$

Of course $\sum_{j\geq 1}\gamma_1(j) = 1$, by (A.10).

Note that we also have, with (A.10),

$$\gamma_1(j_1) = \sum_{j_1+1}^{\infty} f_1(i_1)\frac{\pi(j_1)}{1-P(i_1)} = \sum_{j_1+1}^{\infty} pq^{i_1-1}(1-q^{i_1-1})\frac{1}{(p)_{i_1}}\frac{pq^{j_1-1}}{1-q^{i_1-1}} = pq^{j_1-1}\left[\frac{1}{(p)_\infty} - \frac{1}{(p)_{j_1}}\right].$$



$\circ$: $f_1(i)$
line : $f_{\text{stationary}}(i)$

Figure 2: Comparison between $f_1(i)$ and $f_{\text{stationary}}(i)$, $q = 0.7$

The distribution of the second descent (I_2, J_2) is given by the square of the Markov matrix, i.e.

$$\gamma_2(i_2, j_2) = \sum_{j_0 \geq 1} pq^{j_0-1} \sum_{j_1} \mathbb{P}[J_1 = j_1 | J_0 = j_0] \mathbb{P}[I_2 = i_2, J_2 = j_2 | J_1 = j_1],$$

The distribution of the next descents is related to successive powers of the transition matrix.

A compact form for $\gamma_2(i_2, j_2)$ is given below.

Now, the distribution of the initial value of the *second* descent I_2 is given by (we use (A.3), (A.9))

$$f_2(i_2) := \mathbb{P}[I_2 = i_2] = \sum_{i_1=2}^{\infty} f_1(i_1)\mathbb{P}[I_2 = i_2 | I_1 = i_1] = \sum_{i_1=2}^{\infty} pq^{i_1-1}(1-q^{i_1-1})\frac{1}{(p)_{i_1}}\mathbb{P}[I_2 = i_2 | I_1 = i_1]$$

$$= \sum_{i_1>i_2} pq^{i_1-1}(1-q^{i_1-1})\frac{1}{(p)_{i_1}}\frac{pq^{i_2-1}}{1-q^{i_1-1}}(1-q^{i_2-1})$$

$$+ \sum_{i_1\geq 2} pq^{i_1-1}(1-q^{i_1-1})\frac{1}{(p)_{i_1}}\sum_{j_1<i_1, j_1\leq i_2}\frac{pq^{j_1-1}}{1-q^{i_1-1}}\frac{(p)_{j_1-1}}{(p)_{i_2}}pq^{i_2-1}(1-q^{i_2-1})$$

$$= pq^{i_2-1}(1-q^{i_2-1})\frac{1}{(p)_{i_2-1}(1-pq^{i_2-1})}\left[\frac{1}{(pq^{i_2})_\infty}-1\right]$$

$$+ pq^{i_2-1}(1-q^{i_2-1})\frac{1}{(p)_{i_2}}\sum_{j_1=1}^{i_2} pq^{j_1-1}(p)_{j_1-1}\sum_{i_1>j_1} pq^{i_1-1}\frac{1}{(p)_{i_1}}$$

$$= pq^{i_2-1}(1-q^{i_2-1})\frac{1}{(p)_{i_2-1}(1-pq^{i_2-1})}\left[\frac{1}{(pq^{i_2})_\infty}-1\right]$$

$$+ pq^{i_2-1}(1-q^{i_2-1})\frac{1}{(p)_{i_2}}\sum_{j_1=1}^{i_2} pq^{j_1-1}\frac{1}{1-pq^{j_1-1}}\left[\frac{1}{(pq^{j_1})_\infty}-1\right]$$

$$= pq^{i_2-1}(1-q^{i_2-1})\frac{1}{(p)_{i_2}}\left[\frac{1}{(pq^{i_2})_\infty}-1\right]$$

$$+ pq^{i_2-1}(1-q^{i_2-1})\frac{1}{(p)_{i_2}}\sum_{j_1=1}^{i_2} pq^{j_1-1}\left[\frac{1}{(pq^{j_1-1})_\infty}-\frac{1}{1-pq^{j_1-1}}\right]$$

$$= pq^{i_2-1}(1-q^{i_2-1})\left[\frac{1}{(p)_\infty}-\frac{1}{(p)_{i_2}}\right]$$

$$+ p^2q^{i_2-1}(1-q^{i_2-1})\frac{1}{(p)_{i_2}}\frac{1}{(p)_\infty}\sum_{j_1=0}^{i_2-1} q^{j_1}(p)_{j_1} - p^2q^{i_2-1}(1-q^{i_2-1})\frac{1}{(p)_{i_2}}\sum_{j_1=0}^{i_2-1}\frac{q^{j_1}}{1-pq^{j_1}}$$

$$= pq^{i_2-1}(1-q^{i_2-1})\left[\frac{1}{(p)_\infty}-\frac{1}{(p)_{i_2}}\right]$$

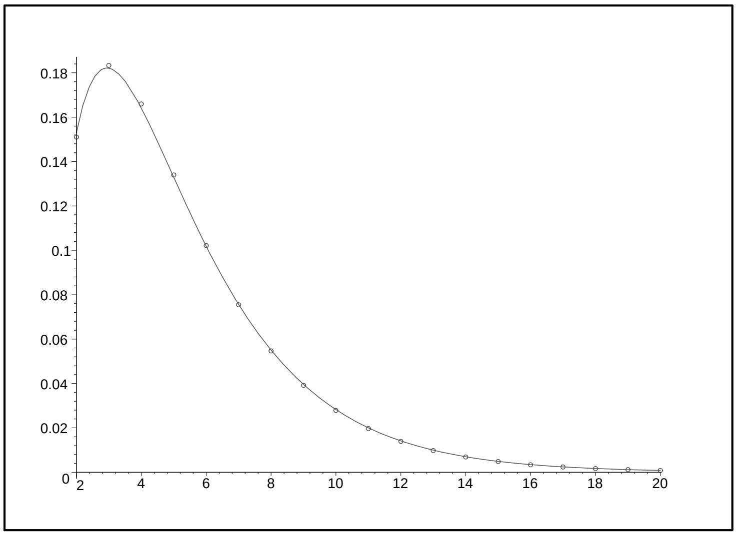
$$+ pq^{i_2-1}(1-q^{i_2-1})\frac{1}{(p)_{i_2}}\frac{1}{(p)_\infty}[1-(p)_{i_2}] - p^2q^{i_2-1}(1-q^{i_2-1})\frac{1}{(p)_{i_2}}\sum_{j_1=0}^{i_2-1}\frac{q^{j_1}}{1-pq^{j_1}}$$

$$= pq^{i_2-1}(1-q^{i_2-1})\frac{1}{(p)_{i_2}}\left[\frac{1}{(p)_\infty}-1\right] - p^2q^{i_2-1}(1-q^{i_2-1})\frac{1}{(p)_{i_2}}\sum_{j_1=0}^{i_2-1}\frac{q^{j_1}}{1-pq^{j_1}}.$$

Note that this gives an explicit expression for $\gamma_2(i_2, j_2)$:

$$\gamma_2(i_2, j_2) = f_2(i_2)\frac{\pi(j_2)}{1-P(i_2)}.$$

A comparison between $f_2(i)$ and $f_{\text{stationary}}(i)$, $q = 0.7$ is given in Figure 3. The convergence to the stationary distribution is quite fast.



$\circ$: $f_2(i)$
line : $f_{\text{stationary}}(i)$

Figure 3: Comparison between $f_2(i)$ and $f_{\text{stationary}}(i)$, $q = 0.7$

Let us check that $\sum_{i=2}^{\infty} f_2(i) = 1$. We have, using (A.8) and (A.1)

$$\sum_{i=2}^{\infty}\left[pq^{i-1}(1-q^{i-1})\frac{1}{(p)_i}\left[\frac{1}{(p)_\infty}-1\right]-p^2q^{i-1}(1-q^{i-1})\frac{1}{(p)_i}\sum_{j_1=0}^{i-1}\frac{q^{j_1}}{1-pq^{j_1}}\right]$$
$$=\frac{1}{(p)_\infty}-1-\sum_{j_1\geq 1}p^2\frac{q^{j_1}}{1-pq^{j_1}}\sum_{i\geq j_1+1}q^{i-1}(1-q^{i-1})\frac{1}{(p)_i}-p^2\frac{1}{1-p}\sum_{i\geq 2}q^{i-1}(1-q^{i-1})\frac{1}{(p)_i}$$
$$=\frac{1}{(p)_\infty}-1-\sum_{j_1\geq 1}p^2\frac{q^{j_1}}{1-pq^{j_1}}\frac{q^{j_1}}{p(p)_{j_1}}-p^2\frac{1}{1-p}\frac{1}{p}=1.$$

8 The moments of descent parameters

The first moments of the first and second descents initial values are analyzed here by intensive use of some combinatorial identities.

We derive the mean of the initial value of the first descent I_1,

$$\begin{aligned}\mathbb{E}(I_1) &= \sum_{i=2}^{\infty} f_1(i)i = \sum_{i\geq 2} pq^{i-1}(1-q^{i-1})\frac{1}{(p)_i}i \\ &= p\sum_{i\geq 1}\frac{q^{i-1}i}{(pq)_i}-p\sum_{i\geq 1}\frac{q^{2i-1}i}{(pq)_i}+p\sum_{i\geq 1}\frac{q^i}{(p)_{i+1}}(1-q^i) \qquad (9) \\ &= p\sum_{i\geq 1}\frac{q^{i-1}i}{(pq)_i}-p\sum_{i\geq 1}\frac{q^{2i-1}i}{(pq)_i}+1 \text{ by (A.4).}\end{aligned}$$

This is Theorem 3 in [5]. However, we want to show now independently that this coincides with closed form obtained earlier as (1).

$$\begin{aligned}
\mathbb{E}(I_1) &= p\sum_{i\ge 0}\frac{q^i}{(p)_{i+1}}(1-q^i)(i+1)\\
&= p\sum_{i\ge 0}\sum_{h=0}^{i}\frac{q^i}{(p)_{i+1}}(1-q^i)\\
&= \frac{p}{q}\sum_{h\ge 0}\sum_{i\ge h}\frac{q^i}{(pq)_i}-\frac{p}{q}\sum_{h\ge 0}\sum_{i\ge h}\frac{q^{2i}}{(pq)_i}\\
&= \frac{p}{q}\sum_{h\ge 0}\frac{q^h}{(pq)_h}\sum_{i\ge 0}\frac{q^i}{(pq^{h+1})_i}-\frac{p}{q}\sum_{h\ge 0}\frac{q^{2h}}{(pq)_h}\sum_{i\ge 0}\frac{q^{2i}}{(pq^{h+1})_i}.
\end{aligned}$$

Now use (A.3):

$$\sum_{i\ge 0}\frac{q^i}{(pq^{h+1})_i}=\frac{1}{pq^h(pq^{h+1})_\infty}+1-\frac{1}{pq^h}.$$

We need also this (use(A.1)):

$$\sum_{i\ge 0}\frac{q^{2i}}{(pq^{h+1})_i}=\frac{1}{pq^{2h}}\frac{1}{(pq^{h+1})_\infty}-\frac{1}{pq^{2h}}-\frac{1}{pq^{h-1}}+1.$$

We now plug these two result in:

$$\begin{aligned}
\mathbb{E}(I_1) &= \frac{p}{q}\sum_{h\ge 0}\frac{q^h}{(pq)_h}\sum_{i\ge 0}\frac{q^i}{(pq^{h+1})_i}-\frac{p}{q}\sum_{h\ge 0}\frac{q^{2h}}{(pq)_h}\sum_{i\ge 0}\frac{q^{2i}}{(pq^{h+1})_i}\\
&= \frac{p}{q}\sum_{h\ge 0}\frac{q^h}{(pq)_h}\left[\frac{1}{pq^h(pq^{h+1})_\infty}+1-\frac{1}{pq^h}\right]-\frac{p}{q}\sum_{h\ge 0}\frac{q^{2h}}{(pq)_h}\left[\frac{1}{pq^{2h}}\frac{1}{(pq^{h+1})_\infty}-\frac{1}{pq^{2h}}-\frac{1}{pq^{h-1}}+1\right]\\
&= \frac{p}{q}\sum_{h\ge 0}\frac{1}{(pq)_h}\left[\frac{1}{p(pq^{h+1})_\infty}+q^h-\frac{1}{p}\right]-\frac{p}{q}\sum_{h\ge 0}\frac{1}{(pq)_h}\left[\frac{1}{p(pq^{h+1})_\infty}-\frac{1}{p}-\frac{q^{h+1}}{p}+q^{2h}\right]\\
&= \frac{p}{q}\sum_{h\ge 0}\frac{1}{(pq)_h}\left[q^h+\frac{q^{h+1}}{p}-q^{2h}\right]\\
&= \frac{p}{q}\sum_{h\ge 0}\frac{1}{(pq)_h}\left[q^h-q^{2h}\right]+\sum_{h\ge 0}\frac{q^h}{(pq)_h}\\
&= \frac{p}{q}\sum_{h\ge 0}\frac{q^h}{(pq)_h}-\frac{p}{q}\sum_{h\ge 0}\frac{q^{2h}}{(pq)_h}+\sum_{h\ge 0}\frac{q^h}{(pq)_h}\\
&= \frac{1}{q}\sum_{h\ge 0}\frac{q^h}{(pq)_h}-\frac{p}{q}\sum_{h\ge 0}\frac{q^{2h}}{(pq)_h}\\
&= \frac{1}{q}\left[\frac{1}{p(pq)_\infty}+1-\frac{1}{p}\right]-\frac{p}{q}\left[\frac{1}{p}\frac{1}{(pq)_\infty}-\frac{1}{p}-\frac{q}{p}+1\right]
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{q}\left[\frac{q}{p(pq)_\infty} - \frac{q}{p} + 2q\right] \\
&= \frac{1}{p(pq)_\infty} - \frac{1}{p} + 2.
\end{aligned}
$$

Since
$$\frac{1}{p(pq)_\infty} - \frac{1}{p} + 2 = 1 - \frac{q}{p} + \frac{q}{p(p)_\infty},$$
we established the formula (1)
$$\mathbb{E}(I_1) = 1 + \frac{q}{p}\Big(-1 + \frac{1}{(p)_\infty}\Big).$$

More generally higher moments are given by
$$\mathbb{E}(I_1^k) = \sum_{h\ge 1} pq^h(1-q^h)(h+1)^k + \sum_{i_1=2}^{\infty}\sum_{j_0=1}^{i_1} pq^{j_0-1}(p)_{j_0-1}\frac{pq^{i_1-1}}{(p)_{i_1}}(1-q^{i_1-1})i_1^k.$$

Let us look at the modified second moment of I_1 (again we use (A.3), (A.1) for simplifications):

$$
\begin{aligned}
\mathbb{EE} &= \sum_{i=2}^{\infty} f_1(i)i(i+1)/2 = p\sum_{i\ge 0}\frac{q^i}{(p)_{i+1}}(1-q^i)(i+1)(i+2)/2 \\
&= p\sum_{0\le k\le h}\sum_{i\ge h}\frac{q^i}{(p)_{i+1}}(1-q^i) \\
&= \frac{1}{q}\sum_{0\le k\le h}\frac{q^h}{(pq)_h} - \frac{p}{q}\sum_{0\le k\le h}\frac{q^{2h}}{(pq)_h} \\
&= \frac{1}{q}\sum_{k\ge 0}\frac{q^k}{(pq)_k}\sum_{h\ge 0}\frac{q^h}{(pq^{k+1})_h} - \frac{p}{q}\sum_{k\ge 0}\frac{q^{2k}}{(pq)_k}\sum_{h\ge 0}\frac{q^{2h}}{(pq^{k+1})_h} \\
&= \frac{1}{q}\sum_{k\ge 0}\frac{q^k}{(pq)_k}\left[\frac{1}{pq^k(pq^{k+1})_\infty} + 1 - \frac{1}{pq^k}\right] - \frac{p}{q}\sum_{k\ge 0}\frac{q^{2k}}{(pq)_k}\left[\frac{1}{pq^{2k}}\frac{1}{(pq^{k+1})_\infty} - \frac{1}{pq^{2k}} - \frac{1}{pq^{k-1}} + 1\right] \\
&= \frac{1}{q}\sum_{k\ge 0}\frac{1}{(pq)_k}\left[\frac{1}{p(pq^{k+1})_\infty} + q^k - \frac{1}{p}\right] - \frac{1}{q}\sum_{k\ge 0}\frac{1}{(pq)_k}\left[\frac{1}{(pq^{k+1})_\infty} - 1 - q^{k+1} + pq^{2k}\right] \\
&= \frac{1}{q}\sum_{k\ge 0}\frac{1}{(pq)_k}\left[\frac{1}{p(pq^{k+1})_\infty} + q^k - \frac{1}{p} - \frac{1}{(pq^{k+1})_\infty} + 1 + q^{k+1} - pq^{2k}\right] \\
&= \frac{1}{q}\sum_{k\ge 0}\frac{1}{(pq)_k}\left[\frac{q}{p(pq^{k+1})_\infty} - \frac{q}{p} + q^k(1+q) - pq^{2k}\right] \\
&= \frac{1}{p}\sum_{k\ge 0}\frac{1}{(pq)_k}\left[\frac{1}{(pq^{k+1})_\infty} - 1\right] + \frac{1+q}{q}\sum_{k\ge 0}\frac{q^k}{(pq)_k} - \frac{p}{q}\sum_{k\ge 0}\frac{q^{2k}}{(pq)_k}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{p}\sum_{k\geq 0}\left[\frac{1}{(pq)_\infty} - \frac{1}{(pq)_k}\right] + \frac{1+q}{q}\left[\frac{1}{p(pq)_\infty} + 1 - \frac{1}{p}\right] - \frac{p}{q}\left[\frac{1}{p}\frac{1}{(pq)_\infty} - \frac{1}{p} - \frac{q}{p} + 1\right] \\
&= \frac{1}{p}\sum_{k\geq 0}\left[\frac{1}{(pq)_\infty} - \frac{1}{(pq)_k}\right] + \frac{2}{p(pq)_\infty} + 3 - \frac{2}{p} \\
&= \frac{1}{p}\lim_{t\to 1}\left[\frac{1}{(pq)_\infty}\frac{1}{1-t} - \sum_{k\geq 0}\frac{t^k}{(pq)_k}\right] + \frac{2}{p(pq)_\infty} + 3 - \frac{2}{p}.
\end{aligned}
$$

To compute this limit, we use (A.7):

$$
\begin{aligned}
&\lim_{t\to 1}\left[\frac{1}{(pq)_\infty}\frac{1}{1-t} - \sum_{k\geq 0}\frac{t^k}{(pq)_k}\right] \\
&= \lim_{t\to 1}\left[\frac{1}{(pq)_\infty}\frac{1}{1-t} - \frac{(q)_\infty}{(pq)_\infty(t)_\infty}\sum_{m\geq 0}\frac{(p)_m(t)_m q^m}{(q)_m}\right] \\
&= \lim_{t\to 1}\left[\frac{1}{(pq)_\infty}\frac{1}{1-t} - \frac{(q)_\infty}{(pq)_\infty(t)_\infty} - \frac{(q)_\infty}{(pq)_\infty(t)_\infty}\sum_{m\geq 1}\frac{(p)_m(t)_m q^m}{(q)_m}\right] \\
&= \lim_{t\to 1}\left[\frac{1}{(pq)_\infty}\frac{1}{1-t} - \frac{(q)_\infty}{(pq)_\infty(1-t)(qt)_\infty}\right] - \frac{(q)_\infty}{(pq)_\infty(q)_\infty}\sum_{m\geq 1}\frac{(p)_m(q)_{m-1}q^m}{(q)_m} \\
&= \frac{1}{(pq)_\infty}\lim_{t\to 1}\left[\frac{1}{1-t} - \frac{(q)_\infty}{(1-t)(qt)_\infty}\right] - \frac{1}{(pq)_\infty}\sum_{m\geq 1}\frac{(p)_m(q)_{m-1}q^m}{(q)_m} \\
&= \frac{1}{(pq)_\infty}\lim_{t\to 1}\frac{(qt)_\infty - (q)_\infty}{(1-t)(qt)_\infty} - \frac{1}{(pq)_\infty}\sum_{m\geq 1}\frac{(p)_m(q)_{m-1}q^m}{(q)_m} \\
&= \frac{1}{(pq)_\infty}\lim_{t\to 1}\frac{\frac{d}{dt}(qt)_\infty\big|_{t=1}(t-1)}{(1-t)(qt)_\infty} - \frac{1}{(pq)_\infty}\sum_{m\geq 1}\frac{(p)_m(q)_{m-1}q^m}{(q)_m} \\
&= -\frac{1}{(pq)_\infty(q)_\infty}\frac{d}{dt}(qt)_\infty\big|_{t=1} - \frac{1}{(pq)_\infty}\sum_{m\geq 1}\frac{(p)_m(q)_{m-1}q^m}{(q)_m} \\
&= \frac{1}{(pq)_\infty}\sum_{k\geq 1}\frac{q^k}{1-q^k} - \frac{1}{(pq)_\infty}\sum_{m\geq 1}\frac{(p)_m q^m}{1-q^m} \\
&= \frac{1}{(pq)_\infty}\sum_{k\geq 1}\frac{[1-(p)_k]q^k}{1-q^k}.
\end{aligned}
$$

So finally:

$$
\mathbb{EE} = \frac{1}{p(pq)_\infty}\sum_{k\geq 1}\frac{[1-(p)_k]q^k}{1-q^k} + \frac{2}{p(pq)_\infty} + 3 - \frac{2}{p}.
$$

From this, the variance can we stated as:

$$
2\mathbb{EE} - 2\mathbb{E}(I_1) - \mathbb{E}^2(I_1).
$$

Now we turn to the second descent. The mean $\mathbb{E}(I_2)$ of the initial value of the second descent is given by (we use (1))

$$\begin{aligned}
\mathbb{E}(I_2) = \sum_{i=2}^{\infty} f_2(i) i &= p\left[\frac{1}{(p)_\infty} - 1\right] \sum_{i\geq 1} i q^{i-1}(1-q^{i-1})\frac{1}{(p)_i} - \sum_{i\geq 1} i p^2 q^{i-1}(1-q^{i-1})\frac{1}{(p)_i}\sum_{j=0}^{i-1}\frac{q^j}{1-pq^j} \\
&= p\left[\frac{1}{(p)_\infty} - 1\right]\sum_{i\geq 0}(i+1)q^i(1-q^i)\frac{1}{(p)_{i+1}} - p^2\sum_{j\geq 0}\frac{q^j}{1-pq^j}\sum_{i\geq j}(i+1)q^i(1-q^i)\frac{1}{(p)_{i+1}} \\
&= \left[\frac{1}{(p)_\infty} - 1\right]\left[1 - \frac{q}{p} + \frac{q}{p(p)_\infty}\right] - p\sum_{j\geq 0}\frac{pq^j}{1-pq^j}\sum_{i\geq j}(i+1)q^i(1-q^i)\frac{1}{(p)_{i+1}} \\
&= \left[\frac{1}{(p)_\infty} - 1 - \sum_{j\geq 0}\frac{pq^j}{1-pq^j}\right]\left[1 - \frac{q}{p} + \frac{q}{p(p)_\infty}\right] + p\sum_{j\geq 0}\frac{pq^j}{1-pq^j}\sum_{0\leq i<j}(i+1)q^i(1-q^i)\frac{1}{(p)_{i+1}}.
\end{aligned}$$

The last inner sum can be computed:

$$\begin{aligned}
S_0 := \sum_{0\leq i<j}(i+1)q^i(1-q^i)\frac{1}{(p)_{i+1}} &= \sum_{0\leq i<j}\sum_{0\leq h\leq i} q^i(1-q^i)\frac{1}{(p)_{i+1}} \\
&= \sum_{0\leq h\leq i<j}\frac{q^i}{(p)_{i+1}} - \sum_{0\leq h\leq i<j}\frac{q^{2i}}{(p)_{i+1}} \\
&= \sum_{0\leq h<j}\sum_{0\leq i<j-h}\frac{q^{i+h}}{(p)_{i+h+1}} - \sum_{0\leq h<j}\sum_{0\leq i<j-h}\frac{q^{2i+2h}}{(p)_{i+h+1}} \\
&= \sum_{0\leq h<j}\frac{q^h}{(p)_{h+1}}\sum_{0\leq i<j-h}\frac{q^i}{(pq^{h+1})_i} - \sum_{0\leq h<j}\frac{q^{2h}}{(p)_{h+1}}\sum_{0\leq i<j-h}\frac{q^{2i}}{(pq^{h+1})_i}.
\end{aligned}$$

We do two auxiliary calculation (with (A.3)):

$$\begin{aligned}
S_1 &:= \sum_{0\leq i<I}\frac{q^i}{(pq^{h+1})_i} \\
&= \sum_{i\geq 0}\frac{q^i}{(pq^{h+1})_i} - \sum_{i\geq I}\frac{q^i}{(pq^{h+1})_i} \\
&= \frac{1}{pq^h(pq^{h+1})_\infty} + 1 - \frac{1}{pq^h} - \frac{q^I}{(pq^{h+1})_I}\sum_{i\geq 0}\frac{q^i}{(pq^{h+I+1})_i} \\
&= \frac{1}{pq^h(pq^{h+1})_\infty} + 1 - \frac{1}{pq^h} - \frac{q^I}{(pq^{h+1})_I}\left[\frac{1}{pq^{h+I}(pq^{h+I+1})_\infty} + 1 - \frac{1}{pq^{h+I}}\right] \\
&= 1 - \frac{1}{pq^h} - \frac{q^I}{(pq^{h+1})_I} + \frac{1}{pq^h(pq^{h+1})_I},
\end{aligned}$$

and with (A.1):

$$
\begin{aligned}
S_2 &:= \sum_{0\le i<I} \frac{q^{2i}}{(pq^{h+1})_i} \\
&= \sum_{i\ge 0} \frac{q^{2i}}{(pq^{h+1})_i} - \sum_{i\ge I} \frac{q^{2i}}{(pq^{h+1})_i} \\
&= \frac{1}{pq^{2h}(pq^{h+1})_\infty} - \frac{1}{pq^{2h}} - \frac{1}{pq^{h-1}} + 1 - \frac{q^{2I}}{(pq^{h+1})_I} \sum_{i\ge 0} \frac{q^{2i}}{(pq^{h+I+1})_i} \\
&= \frac{1}{pq^{2h}(pq^{h+1})_\infty} - \frac{1}{pq^{2h}} - \frac{1}{pq^{h-1}} + 1 - \frac{q^{2I}}{(pq^{h+1})_I} \left[\frac{1}{pq^{2h+2I}(pq^{h+I+1})_\infty} - \frac{1}{pq^{2h+2I}} - \frac{1}{pq^{h+I-1}} + 1 \right] \\
&= -\frac{1}{pq^{2h}} - \frac{1}{pq^{h-1}} + 1 + \frac{1}{pq^{2h}(pq^{h+1})_I} + \frac{q^I}{pq^{h-1}(pq^{h+1})_I} - \frac{q^{2I}}{(pq^{h+1})_I}.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
S_0 &= \sum_{0\le h<j} \frac{q^h}{(p)_{h+1}} \sum_{0\le i<j-h} \frac{q^i}{(pq^{h+1})_i} - \sum_{0\le h<j} \frac{q^{2h}}{(p)_{h+1}} \sum_{0\le i<j-h} \frac{q^{2i}}{(pq^{h+1})_i} \\
&= \sum_{0\le h<j} \frac{q^h}{(p)_{h+1}} \left[1 - \frac{1}{pq^h} - \frac{q^{j-h}}{(pq^{h+1})_{j-h}} + \frac{1}{pq^h(pq^{h+1})_{j-h}} \right] \\
&\quad - \sum_{0\le h<j} \frac{q^{2h}}{(p)_{h+1}} \left[-\frac{1}{pq^{2h}} - \frac{1}{pq^{h-1}} + 1 + \frac{1}{pq^{2h}(pq^{h+1})_{j-h}} + \frac{q^{j-h}}{pq^{h-1}(pq^{h+1})_{j-h}} - \frac{q^{2(j-h)}}{(pq^{h+1})_{j-h}} \right] \\
&= \sum_{0\le h<j} \left[\frac{q^h}{(p)_{h+1}} - \frac{1}{p(p)_{h+1}} - \frac{q^j}{(p)_{j+1}} + \frac{1}{p(p)_{j+1}} \right] \\
&\quad - \sum_{0\le h<j} \left[-\frac{1}{p(p)_{h+1}} - \frac{q^{h+1}}{p(p)_{h+1}} + \frac{q^{2h}}{(p)_{h+1}} + \frac{1}{p(p)_{j+1}} + \frac{q^{j+1}}{p(p)_{j+1}} - \frac{q^{2j}}{(p)_{j+1}} \right] \\
&= \sum_{0\le h<j} \left[\frac{q^h}{p(p)_{h+1}} - \frac{q^j}{p(p)_{j+1}} - \frac{q^{2h}}{(p)_{h+1}} + \frac{q^{2j}}{(p)_{j+1}} \right].
\end{aligned}
$$

Recalling the expressions for S_1 and S_2 that we just derived, we get

$$
\begin{aligned}
S_0 &= \sum_{0\le h<j} \left[\frac{q^h}{pq(pq)_h} - \frac{q^{2h}}{q(pq)_h} \right] - \frac{jq^j}{p(p)_{j+1}} + \frac{jq^{2j}}{(p)_{j+1}} \\
&= \frac{1}{pq}\left[1 - \frac{1}{p} - \frac{q^j}{(pq)_j} + \frac{1}{p(pq)_j} \right] - \frac{1}{q}\left[-\frac{2q}{p} + \frac{1}{p(pq)_j} + \frac{q^{j+1}}{p(pq)_j} - \frac{q^{2j}}{(p)_j} \right] - \frac{jq^j}{p(p)_{j+1}} + \frac{jq^{2j}}{(p)_{j+1}} \\
&= \frac{1}{pq} - \frac{1}{p^2q} - \frac{q^j}{pq(pq)_j} + \frac{1}{p^2q(pq)_j} + \frac{2}{p} - \frac{1}{pq(pq)_j} - \frac{q^j}{p(pq)_j} + \frac{q^{2j}}{q(p)_j} - \frac{jq^j}{p(p)_{j+1}} + \frac{jq^{2j}}{(p)_{j+1}} \\
&= \frac{2}{p} - \frac{1}{p^2} + \frac{q}{p^2(p)_{j+1}} - \frac{(1+q)q^j}{pq(p)_{j+1}} + \frac{q^{2j}}{q(p)_j} - \frac{jq^j}{p(p)_{j+1}} + \frac{jq^{2j}}{(p)_{j+1}}.
\end{aligned}
$$

Therefore we found a first expression for the mean:

$$
\begin{aligned}
\mathbb{E}(I_2) &= \frac{1}{p}\left[\frac{1}{(p)_\infty} - 1 - \sum_{j\ge 0}\frac{pq^j}{1-pq^j}\right]\left[2p-1+\frac{q}{(p)_\infty}\right] \\
&\quad + \frac{1}{p}\sum_{j\ge 0}\frac{pq^j}{1-pq^j}\left[2p-1+\frac{q}{(p)_{j+1}} - \frac{p(1+q)q^j}{q(p)_{j+1}} + \frac{p^2q^{2j}}{q(p)_j} - \frac{pjq^j}{(p)_{j+1}} + \frac{jp^2q^{2j}}{(p)_{j+1}}\right] \\
&= \frac{1}{p}\left[\frac{1}{(p)_\infty} - 1\right]\left[2p-1+\frac{q}{(p)_\infty}\right] \\
&\quad + \frac{1}{p}\sum_{j\ge 0}\frac{pq^j}{1-pq^j}\left[\frac{q}{(p)_{j+1}} - \frac{q}{(p)_\infty} - \frac{p(1+q)q^j}{q(p)_{j+1}} + \frac{p^2q^{2j}}{q(p)_j} - \frac{pjq^j}{(p)_{j+1}} + \frac{jp^2q^{2j}}{(p)_{j+1}}\right] \\
&= \frac{1}{p}\left[\frac{1}{(p)_\infty} - 1\right]\left[2p-1+\frac{q}{(p)_\infty}\right] \\
&\quad + \frac{1}{p}\sum_{j\ge 0}\frac{pq^j}{1-pq^j}\left[\frac{q}{(p)_{j+1}} - \frac{q}{(p)_\infty} - \frac{p(1+q)q^j}{q(p)_{j+1}} - \frac{pjq^j}{(p)_{j+1}} + \frac{jp^2q^{2j}}{(p)_{j+1}}\right] + \frac{1}{p}\sum_{j\ge 0}\frac{p^3q^{3j}}{q(p)_{j+1}}.
\end{aligned}
$$

The last sum can be simplified; we will use Heine with $t = pq^h$, $a = 0$, $c = q$, $b = q$:

$$
\begin{aligned}
\sum_{i\ge 0}\frac{q^{3i}}{(pq^{h+1})_i} &= \frac{(q)_2}{(pq^{h+1})_\infty}\sum_{m\ge 0}\frac{(pq^h)_m(q^3)_m}{(q)_m}q^m \\
&= \frac{(q)_2}{(pq^{h+1})_\infty}\frac{(pq^h)_\infty(q^4)_\infty}{(q)_\infty}\sum_{m\ge 0}\frac{(q)_m}{(q)_m(q^4)_m}(pq^h)^m \\
&= \frac{1-pq^h}{1-q^3}\sum_{m\ge 0}\frac{1}{(q^4)_m}(pq^h)^m \\
&= \frac{(1-pq^h)(q)_3}{(pq^h)^3}\sum_{m\ge 3}\frac{1}{(q)_m}(pq^h)^m \\
&= \frac{(1-pq^h)(q)_3}{(pq^h)^3}\left[\frac{1}{(pq^h)_\infty} - 1 - q^h - \frac{(pq^h)^2}{(q)_2}\right] \\
&= \frac{(q)_3}{(pq^h)^3}\frac{1}{(pq^{h+1})_\infty} - \frac{(1-pq^h)(q)_3}{p^3q^{3h}} - \frac{(1-pq^h)(q)_3}{p^3q^{2h}} - \frac{(1-pq^h)(1-q^3)}{pq^h}.
\end{aligned}
$$

From this we derive (set $h = -1$)

$$
\begin{aligned}
\sum_{j\ge 0}\frac{p^3q^{3j}}{q(p)_{j+1}} &= \frac{p^3}{q}\left[\frac{(q)_3}{(p/q)^3}\frac{1}{(p)_\infty} - \frac{(1-p/q)(q)_3}{p^3q^{-3}} - \frac{(1-p/q)(q)_3}{p^3q^{-2}} - \frac{(1-p/q)(1-q^3)}{p/q}\right] \\
&= \frac{p^3}{q}\left[\frac{(q)_3q^3}{p^3}\frac{1}{(p)_\infty} - \frac{q^2(q-p)(q)_3}{p^3} - \frac{q(q-p)(q)_3}{p^3} - \frac{(q-p)(1-q^3)}{p}\right]
\end{aligned}
$$

$$= \frac{(q)_3 q^2}{(p)_\infty} - q(q-p)(q)_3 - (q-p)(q)_3 - \frac{p^2(q-p)(1-q^3)}{q}.$$

Hence

$$\begin{aligned}
\mathbb{E}(I_2) &= \frac{1}{p}\left[\frac{1}{(p)_\infty} - 1\right]\left[2p - 1 + \frac{q}{(p)_\infty}\right] \\
&\quad + \frac{1}{p}\sum_{j\ge 0} \frac{pq^j}{1-pq^j}\left[\frac{q}{(p)_{j+1}} - \frac{q}{(p)_\infty} - \frac{p(1+q)q^j}{q(p)_{j+1}}\right] - \frac{1}{p}\sum_{j\ge 0}\frac{jp^2q^{2j}}{(p)_{j+1}} \\
&\quad + \frac{1}{p}\left[\frac{(q)_3 q^2}{(p)_\infty} - q(q-p)(q)_3 - (q-p)(q)_3 - \frac{p^2(q-p)(1-q^3)}{q}\right].
\end{aligned}$$

We still need (we use S_2)

$$\begin{aligned}
\sum_{j\ge 0}\frac{jq^{2j}}{(p)_{j+1}} &= \sum_{h\ge 0}\sum_{j\ge h}\frac{q^{2j}}{(p)_{j+1}} \\
&= \sum_{h\ge 0}\frac{q^{2h}}{(p)_{h+1}}\sum_{j\ge 0}\frac{q^{2j}}{(pq^{h+1})_j} \\
&= \sum_{h\ge 0}\frac{q^{2h}}{(p)_{h+1}}\left[-\frac{1}{pq^{2h}} - \frac{1}{pq^{h-1}} + 1 + \frac{1}{pq^{2h}(pq^{h+1})_\infty}\right] \\
&= -\frac{1}{p}\sum_{h\ge 0}\frac{1}{(p)_{h+1}}\left[1 - \frac{1}{(pq^{h+1})_\infty}\right] - \frac{q}{p}\sum_{h\ge 0}\frac{q^h}{(p)_{h+1}} + \sum_{h\ge 0}\frac{q^{2h}}{(p)_{h+1}} \\
&= \frac{1}{pq}\sum_{h\ge 0}\left[\frac{1}{(pq)_\infty} - \frac{1}{(pq)_h}\right] - \frac{1}{p}\left[\frac{1}{p(pq)_\infty} + 1 - \frac{1}{p}\right] + \frac{1}{q}\left[\frac{1}{p(pq)_\infty} - \frac{1}{p} - \frac{q}{p} + 1\right] \\
&= \frac{1}{pq}\frac{1}{(pq)_\infty}\sum_{k\ge 1}\frac{[1-(p)_k]q^k}{1-q^k} - \frac{1}{p}\left[\frac{1}{p(pq)_\infty} + 1 - \frac{1}{p}\right] + \frac{1}{q}\left[\frac{1}{p(pq)_\infty} - \frac{1}{p} - \frac{q}{p} + 1\right],
\end{aligned}$$

hence

$$\begin{aligned}
\mathbb{E}(I_2) &= \frac{1}{p}\left[\frac{1}{(p)_\infty} - 1\right]\left[2p - 1 + \frac{q}{(p)_\infty}\right] \\
&\quad + \frac{1}{p}\sum_{j\ge 0}\frac{pq^j}{1-pq^j}\left[\frac{q}{(p)_{j+1}} - \frac{q}{(p)_\infty} - \frac{p(1+q)q^j}{q(p)_{j+1}}\right] \\
&\quad - \frac{1}{p^2q}\frac{1}{(pq)_\infty}\sum_{k\ge 1}\frac{[1-(p)_k]q^k}{1-q^k} + \frac{1}{p^2}\left[\frac{1}{p(pq)_\infty} + 1 - \frac{1}{p}\right] - \frac{1}{pq}\left[\frac{1}{p(pq)_\infty} - \frac{1}{p} - \frac{q}{p} + 1\right] \\
&\quad + \frac{1}{p}\left[\frac{(q)_3 q^2}{(p)_\infty} - q(q-p)(q)_3 - (q-p)(q)_3 - \frac{p^2(q-p)(1-q^3)}{q}\right].
\end{aligned}$$

Although this computation was already quite involved, we could get all cross-moments from (4), but only with considerable effort.

9 Markov Chains and Sojourn times

In this section, we obtain the asymptotic distribution of the number of descents initial values in some interval.

9.1 General asymptotic distribution

Let X_i, $i = 1, \ldots, m$ be an ergodic Markov chain (MC) and A be a subset of states. Assume that the MC is stationary and set $x_i := [\![X_i \in A]\!]$, with $M := \mathbb{E}(x_i)$. The number D of times the MC is in A on $1, \ldots, m$ is such that

$$\begin{aligned}
\mathbb{E}(D) &= mM, \\
\mathbb{V}(D) &= \mathbb{E}\Big[\sum_{i=1}^{m}(x_i - M)^2 + 2\sum_{i=1}^{m-1}\sum_{j=i+1}^{m}(x_i - M)(x_j - M)\Big].
\end{aligned}$$

If

$$\mathbb{E}[(x_i - M)(x_j - M)] = 0, \quad j \geq i + 2,$$

then, setting $B := \mathbb{E}(x_i x_{i+1})$, we obtain

$$\mathbb{V}(D) = mM(1 - M) + 2(m - 1)(B - M^2).$$

By the central limit theorem for MC, we have

$$\frac{D - \mathbb{E}(D)}{\sqrt{\mathbb{V}(D)}} \sim \mathcal{N}(0, 1), \quad m \to \infty.$$

9.2 Number of descents values in some interval

Here, the states of the MC are couples $Y_t Y_{t+1}$ with geometric distribution. Also the MC starts with the stationary distribution $pq^{i-1}pq^{j-1}$, hence $m = n - 1$.

If we are interested in the asymptotic (gaussian) distribution of the number of descents initial values in some interval $[\tau, \tau + \Delta]$, we compute

$$\begin{aligned}
M &= \sum_{i=\tau}^{\tau+\Delta} pq^{i-1}(1 - q^{i-1}), \\
B &= \sum_{i=\tau+1}^{\tau+\Delta} pq^{i-1}\sum_{j=\tau}^{i-1} pq^{j-1}(1 - q^{j-1}).
\end{aligned}$$

10 Permutations

Starting from n GEOM(p) RV, as $q \to 1$, we can derive the asymptotic properties of first and second descents, in a large permutation. We will consider large size ($n \to \infty$) permutations of $\{1, \ldots, n\}$, or n-permutations for short. It is well known that all rank statistics of an n-permutation can be derived from the corresponding ones of a sequence of n GEOM(p) RVs as $q \to 1$. But, of course, it is not possible to deduce the moments of the beginning (initial height) of the first descent of a n-permutation from the corresponding moments of the geometric variables statistic, as $q \to 1$. However, the asymptotic *distribution* of this RV can be derived as follows. Let $q = 1 - \varepsilon$. This gives an asymptotic distribution function for each GEOM RV K ($\varepsilon \to 0$)

$$1 - q^i \sim F(i) = 1 - e^{-i\varepsilon}.$$

Set $U = F(K)$; U is asymptotically distributed as a uniform $[0,1]$ RV. Let us consider n GEOM(p) RVs $K_1, \ldots, K_n$. If we scale the corresponding U variables with n (i.e. multiply by n), take the integer part, we have asymptotically a permutation on n, for large n. More precisely the rank of U_i is the value of the ith element of the permutation. This gives

$$u = 1 - e^{-i\varepsilon}, \quad du = \varepsilon(1-u)di, \quad i = -\ln(1-u)/\varepsilon.$$

Set $g(u) := f_1(i)$ as given by (7). Using n geometric RVs instead of an infinite sequence of ones intoduces only an exponentially small error.

For instance, this leads to

$$\mathbb{E}(I_1) \sim n \int_0^1 g(u) u \frac{du}{\varepsilon(1-u)}, \quad \varepsilon \to 0.$$

We have by simple algebra,

$$g(u) \sim \varepsilon(1-u)u + e^u u(1 - e^{-u})\varepsilon(1-u),$$

and the asymptotic density of I_1/n is given by (we multiply by $di = du/(\varepsilon(1-u))$)

$$h_1(u) = u[1 + e^u(1 - e^{-u})] = ue^u = u \sum_{j=0}^{\infty} \frac{u^j}{j!}.$$

But this has a clear direct probabilistic interpretation: if $U_{j+1} = u$, the probability

$$\mathbb{P}[U_1 \le U_2 \le \cdots \le U_j \le U_{j+1} \ge U_{j+2}] = u\frac{u^j}{j!}.$$

Again, as n is large, we can use an infinite summation on j, with exponentially small error.

Note that

$$\int_0^1 h_1(u)du = 1,$$

as expected, and

$$\mathbb{E}(I_1) \sim n \int_0^1 h_1(u)udu = n(e-2) := nE_{1,0}, \quad n \to \infty,$$

which conforms to Theorem 9 of [5]. All moments can be derived. For instance

$$\mathbb{E}(I_1^2) \sim n^2 \int_0^1 h_1(u)u^2du = n^2(6 - 2e) := n^2 E_{2,0},$$
$$\mathbb{E}(I_1^3) \sim n^3 E_{3,0}, \text{ with } E_{3,0} = 9e - 24.$$

A general expression can be derived as follows. We have (we drop the second index)

$$E_k = e - (k+1)E_{k-1}, \quad E_0 = 1.$$

This first order recursion can be solved by iteration:

$$E_k = (k+1)!(-1)^k \left[e \sum_{i=1}^{k} \frac{(-1)^i}{(i+1)!} + 1 \right]. \tag{10}$$

By convention, $E_{-1} = \int_0^1 e^u du = e - 1$.

We could derive similar expressions for J_2.

The errors terms, as $n \to \infty$ can be derived as follows. The rank R of U_{j+1} is such that $R-(j+2)$ is $\mathsf{Binomial}(n-(j+2), u)$, if U_{j+1} has value u and corresponds to the first descent. Indeed, j values are already below u (on the left) and one on the right. If d values among the $n-(j+2)$ remaining ones are below u, then U_{j+1} possesses the rank $d+j+2$. So the moments of R/n are given by the characteristic function

$$F := \left[1 + u\left(e^{\mathbf{i}\theta/n} - 1\right)\right]^{n-(j+2)} e^{\mathbf{i}\theta(j+2)/n},$$

from which we derive

$$\begin{aligned}
\mathbb{E}(R/n|j) &\sim u + \frac{-u(u-1)(j+2)}{n}, \\
\mathbb{E}(R^2/n^2|j) &\sim u^2 + \frac{-u(u-1)(5+2j)}{n}.
\end{aligned}$$

We multiply by $u\frac{u^j}{j!}$, and sum on j; this gives

$$\begin{aligned}
\sum_{j=0}^{\infty} u\frac{u^j}{j!}\mathbb{E}(R/n|j) &\sim h_1(u)u + \frac{-e^u u(u-1)(u+2)}{n}, \\
\sum_{j=0}^{\infty} u\frac{u^j}{j!}\mathbb{E}(R^2/n^2|j) &\sim h_1(u)u^2 + \frac{-e^u u^2(u-1)(2u+5)}{n}.
\end{aligned}$$

Hence, integrating on $u \in [0..1]$, we finally obtain

$$\begin{aligned}
\mathbb{E}(I_1/n) &\sim E_{1,0} + \frac{e-2}{n}, \\
\mathbb{E}(I_1^2/n^2) &\sim E_{2,0} + \frac{20-7e}{n}.
\end{aligned}$$

The first expression fits with Theorem 9 in [5]; the second moment wasn't computed in this paper.

The joint moments of the first and second descent I_1, I_2 are asymptotically computed as follows. First, from (4),

$$g_1(u_2, u_1) = [\![u_2 < u_1]\!]\varepsilon(1-u_2)\frac{u_2}{u_1} + \int_0^{\min(u_1,u_2)} \frac{dv_1}{\varepsilon(1-v_1)}\varepsilon\frac{1-v_1}{u_1}e^{u_2-v_1}\varepsilon u_2(1-u_2)$$

$k \backslash l$	0	1	2
0	1	$e^2 - e - 4$	$-2e^2 - e + 18$
1	$e - 2$	$-7e/2 + 10$	$20e - 54$
2	$6 - 2e$	$e^2 + 32e/3 - 36$	$-2e^2 - 235e/3 + 228$

Table 1: $E_{k,l}$

$$= [\![u_2 < u_1]\!]\varepsilon(1-u_2)\frac{u_2}{u_1} + e^{u_2}\varepsilon u_2 \frac{1-u_2}{u_1}(1 - e^{-\min(u_1,u_2)}).$$

This leads to the Markov kernel

$$h(u_2, u_1) = \frac{g_1(u_2, u_1)}{\varepsilon(1-u_2)} = [\![u_2 < u_1]\!]\frac{u_2}{u_1} + e^{u_2}u_2\frac{1}{u_1}(1 - e^{-\min(u_1,u_2)}). \tag{11}$$

This can also be derived from the $\mathsf{uniform}[0,1]$ random variables properties.

The joint moment is asymptotically given given by $n^2 E_{k,l}$, with

$$\begin{aligned} E_{k,l} &= \int_0^1 \int_0^1 h_1(u_1)h(u_2,u_1)u_1^k u_2^l du_1 du_2 \\ &= \int_0^1 u_1 e^{u_1} \int_0^1 u_1^k u_2^l \frac{1}{u_1}\left[[\![u_2 < u_1]\!]u_2 + e^{u_2}u_2(1 - e^{-\min(u_1,u_2)})\right] du_1 du_2. \end{aligned}$$

The first values of $E_{k,l}$ are given in Table 1.

The asymptotic density of I_2/n is given by

$$h_2(u) = \int_0^1 h_1(u_1)h(u,u_1)du_1 = u[e^{1+u} - ue^u - e^u] = ue^u[-1 - u + e],$$

and similarly for $h_3(u)$. This gives eventually

$$h_3(u) = \int_0^1 h_2(u_2)h(u_3,u_2)du_2 = ue^u[(u + u^2/2) + e(-u-2) + e^2].$$

The convergence to the stationary distribution $h(u) = 2u$ is very fast. $h(u)$ can be derived from (5), or directly by considering two successive RVs U_k. Figure 4 gives $h(u), h_1(u), h_2(u)$

The difference $h_3(u) - h(u)$ is given in Figure 5.

A general expression for $h_i(u)$ can be computed as follows: we observe that $h_i(u)$ is of the form

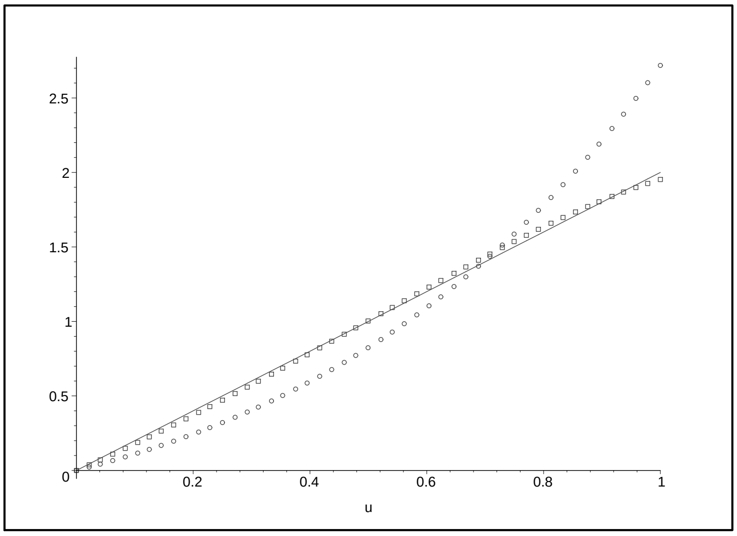
$$h_i(u) = ue^u \sum_{k=0}^{i-1} e^k P_{i,k}(u),$$

where

$$P_{i,k}(u) = \sum_{l=0}^{i-k-1} P(i,k,l)u^l.$$

But, setting $E_l := D_{l,0} + D_{l,1}e$, from (11) and (10),

$$h_i(u) = \int_0^1 h_{i-1}(u)h(u,v)dv$$



$\circ$: $h_1(u)$
$\square$: $h_2(u)$
line : $h(u)$

Figure 4: $h(u), h_1(u), h_2(u)$

$$
\begin{aligned}
&= \int_0^1 h_{i-1}(v)\frac{1}{v}\left[[\![u < v]\!]u + e^u u(1 - e^{-\min(v,u)})\right] dv \\
&= \int_0^u h_{i-1}(v)\frac{1}{v}e^u u(1 - e^{-v})dv + \int_u^1 h_{i-1}(v)\frac{1}{v}e^u u dv \\
&= \int_0^1 h_{i-1}(v)\frac{1}{v}e^u u dv - \int_0^u h_{i-1}(v)\frac{1}{v}e^u u e^{-v} dv \\
&= e^u u\left[\sum_{k=0}^{i-2} e^k \int_0^1 P_{i-1,k}(v)e^v dv - \sum_{k=0}^{i-2} e^k \int_0^u P_{i-1,k}(v)dv\right] \\
&= e^u u\left[\sum_{k=0}^{i-2} e^k \sum_{l=0}^{i-k-2} P_{i-1,k,l}[D_{l-1,0} + D_{l-1,1}e] - \sum_{k=0}^{i-2} e^k \sum_{l=0}^{i-k-2} P_{i-1,k,l}\frac{u^{l+1}}{l+1}\right].
\end{aligned}
$$

This leads to the following recurrence

$$
\begin{aligned}
&P_{i,i-1}(u) = 1, \\
&\quad P_{i,0,0} = \sum_{l=0}^{i-2} P_{i-1,0,l}D_{l-1,0}, \\
&\quad P_{i,k,0} = \sum_{l=0}^{i-k-2} P_{i-1,k,l}D_{l-1,0} + \sum_{l=0}^{i-k-1} P_{i-1,k-1,l}D_{l-1,1}, \quad k = 1, \ldots, i-2, \\
&\quad P_{i,k,l} = -P_{i-1,k,l-1}/l, \quad k = 0, \ldots, i-2,\ l = 1, \ldots, i-k-1.
\end{aligned}
$$

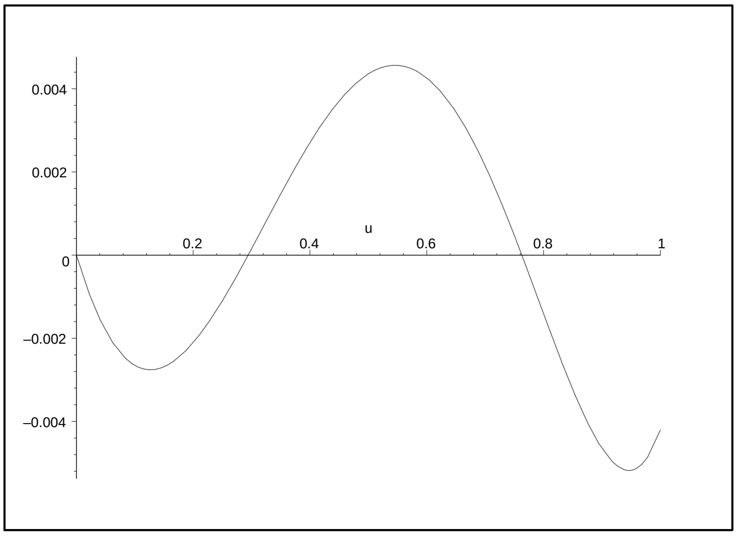


Figure 5: $h_3(u) - h(u)$

Finally, asymptotically, given I_k, J_k is uniform $[1 \ldots I_k - 1]$.

The number of descents values of I/n in some interval $[\tau, \tau + \Delta]$ is now related to

$$
\begin{aligned}
M &= \int_{\tau}^{\tau+\Delta} u du, \\
B &= \int_{\tau}^{\tau+\Delta} du_1 \int_{\tau}^{u_1} u_2 du_2.
\end{aligned}
$$

11 Conclusion

In this paper, we have made a combinatorial and probabilistic study of initial- and endheights of first, second, ... descents in samples of geometrically distributed random variables and in permutations. Several other (similar) models can be analyzed with our tools, let us mention: the weak model and/or ascents with the probabilistic approach, k-descents, size d or more descents, (see [2]), last descents, (see [4]), k-ascents with a combinatorial approach (see [2]), etc.

We leave these topics to future work (or to the interested reader), in order to keep the length of the paper within reasonable limits.

Appendix

A Some combinatorial identities

We will use intensively Heine's formula:

$$\sum_{m\geq 0}\frac{(a)_m(b)_m t^m}{(q)_m(c)_m}=\frac{(b)_\infty(at)_\infty}{(c)_\infty(t)_\infty}\sum_{m\geq 0}\frac{(c/b)_m(t)_m b^m}{(q)_m(at)_m}.$$

Several q-combinatorial relations are deduced from Heine.

$$pz\sum_{m\geq 1}\frac{q^{2m}}{(pzq)_m}=\frac{1}{z}\left[\frac{1}{(pqz)_\infty}-1\right]-q.\text{ This is Thm.1 in [5].}\tag{A.1}$$

$$\sum_{m\geq 0}(pz)_m q^m=\frac{1}{pz}[1-(pz)_\infty].\text{ This is (5) in [5].}\tag{A.2}$$

$$pz\sum_{m\geq 1}\frac{q^m}{(pz)_{m+1}}=\frac{1}{1-pz}\left[\frac{1}{(pqz)_\infty}-1\right].\text{ This is implicitly used in [5].}\tag{A.3}$$

Heine with $t=q$, $a=0$, $b=q$, $c=pzq$, and (A.2).

$$\sum_{m\geq 1}\frac{pq^m(1-q^m)}{(p)_{m+1}}=1.\text{ Application of (A.3), (A.1).}\tag{A.4}$$

$$\sum_{m\geq 0}(pz)_m q^{2m}=\frac{1}{(pz)^2}\left[1-(pz)_\infty-\frac{1-pz}{q}+\frac{(pz)_\infty}{q}\right].\text{ This is (6) in [5].}\tag{A.5}$$

$$\sum_{k\geq 1}\frac{p^k}{(q)_k}=-1+\frac{1}{(p)_\infty}.\text{ Heine with } t=p,\ a=0,\ b=q,\ c=q.\tag{A.6}$$

$$\sum_{k\geq 0}\frac{t^k}{(pq)_k}=\frac{(q)_\infty}{(pq)_\infty(t)_\infty}\sum_{m\geq 1}\frac{(p)_m(t)_m q^m}{(q)_m}.\text{ Heine with } a=0,\ b=q,\ c=pq.\tag{A.7}$$

$$(p)_{j-1}\sum_{i=j}^{\infty}\frac{pq^{i-1}(1-q^{i-1})}{(p)_i}=q^{j-1}.\text{ Application of (A.1), (A.3).}\tag{A.8}$$

$$\begin{aligned}\sum_{0\leq j<J}pq^j(p)_j&=p\sum_{j\geq 0}q^j(p)_j-p\sum_{j\geq J}q^j(p)_j\\&=p\frac{1}{p}\Big[1-(p)_\infty\Big]-pq^J(p)_J\frac{1}{pq^J}\Big[1-(pq^J)_\infty\Big]\\&=1-(p)_\infty-(p)_J\Big[1-(pq^J)_\infty\Big]\\&=1-(p)_J.\text{ By (A.2).}\end{aligned}\tag{A.9}$$

$$\begin{aligned}\sum_{i_1>i_0}\frac{pq^{i_1-1}}{(p)_{i_1}}&=\frac{1}{(p)_{i_0-1}(1-pq^{i_0-1})}\left[\frac{1}{(pq^{i_0})_\infty}-1\right]\\&=\frac{1}{(p)_{i_0}}\left[\frac{1}{(pq^{i_0})_\infty}-1\right]\\&=\left[\frac{1}{(p)_\infty}-\frac{1}{(p)_{i_0}}\right].\text{ By (A.3).}\end{aligned}\tag{A.10}$$

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