THE ASYMMETRIC LEADER ELECTION ALGORITHM: ANOTHER APPROACH

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Abstract. The asymmetric leader election algorithm has obtained quite a bit of attention lately. In this paper we want to analyze the following asymptotic properties of the number of rounds: limiting distribution function, all moments in a simple automatic way, asymptotics for $p \to 0$, $p \to 1$ (where p denotes the "killing" probability). This also leads to a few interesting new identities. We use two paradigms: first, in some urn model, we have asymptotic independence of urns behaviour as far as random variables related to urns with a fixed number of balls are concerned. Next, we use a technique easily leading to the asymptotics of the moments of extreme-value related distribution functions.

1. Introduction

The following leader election algorithm has obtained quite a bit of attention lately:

Starting with n players, each one throws a coin, and the k of them which have thrown head (with probability q) go on, and the leader will be found amongst them, using the same strategy. Should nobody advance, will the party repeat the procedure.

The first paper considering this [11] treated only the symmetric case $p = q = \frac{1}{2}$, but the subsequent writings [2, 5, 6] dealt with the general situation. The most interesting parameter here is the number $K(n)$ of rounds until a leader has been identified.

In our asymptotic analysis (for $n \to \infty$) we are focussing on the *limiting distribution function;* how to compute *all moments,* essentially in an automatic way. Further, we investigate what happens, if the parameter p gets small ($p \to 0$) or large ($p \to 1$). Along the way some interesting identities are encountered. In particular, in the appendix, we find short proofs of identities that identify the previous results of Janson-Szpankowski and Knessl.

To give a flavour of the results, we easily (re)derive that the average number of rounds to elect the leader is given by

$$x_n \sim \log_{1/q}(n) + \frac{U_1 + \gamma - 1}{\ln(1/q)} + \frac{1}{2} + \frac{1}{\ln(1/q)} \sum_{l \neq 0} \left[T_1(\chi_l) - \Gamma(\chi_l)(1+\chi_l)\right] e^{-2l\pi\mathbf{i}\ln n/\ln(1/q)},$$

with the constants

$$U_1 = \sum_{i=2}^{\infty} \frac{p^i x_i}{i}$$

and

$$T_1(\chi_l) := \sum_{i=2}^{\infty} \frac{p^i}{i!} x_i \Gamma(i + \chi_l).$$

(We have $\chi_l = 2\pi\mathbf{i}l/\ln(1/q)$, $\mathbf{i}$ denotes the imaginary unit.) Note that the series in the asymptotic formula represents a periodic function of $\log_{1/q}(n)$ of small amplitude.

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While this formula was already known, the following result about the periodic part of the second moment of the number of rounds to elect the leader is new (the functions and constants are given in Appendix A):

Proposition 1.1.

$$w_2 = \sum_{l\neq 0}\left\{\frac{T_2(\chi_l)}{L} - 2M\frac{T_1(\chi_l)}{L} - 2\frac{T_3(\chi_l)}{L^2} + 2M\frac{R_1(\chi_l)}{L^2} + \frac{R_2(\chi_l)+R_3(\chi_l)}{L^3}\right.$$
$$\left. + \Gamma(1+\chi_l)\left[\frac{2\psi(1+\chi_l)}{L^2} + \frac{1}{L} + 2\frac{M}{L}\right]\right\} e^{-2l\pi\mathbf{i}\log n}.$$

It might look paradoxical at first sight, that the asymptotic formula involves the quantities x_i on the right side. However, this happens often, and convergence is quite good, so that with only a few terms (obtained directly from the recursion) a good approximation of the numerical values can be obtained.

The paper is organized as follows: Section 2 provides the notations we use in the sequel. Section 3 analyses the asymptotic probability distribution of K. Section 4 is devoted to the asymptotics of the moments of $K - \log n^*$. Section 5 considers the cases $p \to 0$, $p \to 1$. Appendix A, provides some definitions and identities which are used in the paper. Appendices B and C prove some equivalences for Knessl's results.

Since this paper depends methodologically on our earlier papers [9] and [8], Section 3 gives a synopsis of these, for the convenience of the reader, to make the current paper independent.

Note that in [10], H. Mohamed uses another interesting probabilistic approach to analyzing the mean cost.

2. Notations

Let the random variable X be geometrically distributed, i. e., $\mathbb{P}(X = j) = pq^{j-1}$, with $q = 1 - p$: We interpret p as the killing probability, and q as the survival probability. Note that, to compare with [5], we must interchange p and q. Let us consider the game as an urn model, with urns labelled $1, 2, \dots$, where we throw n balls, and the probability of each ball falling into urn j being given by pq^{j-1}. The balls at level j represent the candidates who are killed at this level. Let us denote by $K(n)$ the number of rounds, when we start with n players. Note that, when all the players are killed, they are magically resurrected and try again. This increases the parameter K, but leaves the party at the same level (same urn).

Note that we often speak synonymously about players, balls, items, etc., and also about urns, levels, etc.

We drop the n-dependency when there is no ambiguity, to ease the notation. We will use the following notations:

$$\begin{aligned}
\Pi(\lambda, u) &:= e^{-\lambda}\lambda^u/u!, \text{ (Poisson distribution)},\\
n^* &:= n\frac{p}{q},\\
Q &:= 1/q,\\
\log &:= \log_Q,\\
M &:= \log p,\\
\eta &:= j - \log n^*,
\end{aligned}$$

$$\begin{aligned}
L &:= \ln Q,\\
\{x\} &:= \text{ fractional part of } x,\\
\widetilde{\alpha} &:= \alpha/L,\\
\chi_l &:= \frac{2l\pi\mathbf{i}}{L},\\
P(i,k) &:= \mathbb{P}(K=k|\text{ we start with } i \text{ candidates}),\\
G(i,k) &:= \sum_{u=0}^{k} P(i,u),\\
x_i &:= \mathbb{E}(K(i)) = \sum_{k=1}^{\infty} P(i,k)k,\\
x_i^{(2)} &:= \mathbb{E}(K(i)^2) = \sum_{k=1}^{\infty} P(i,k)k^2.
\end{aligned}$$

We have the special values $P(1,0)=1$, $P(1,k)=0$, $k>0$, $P(i,0)=0$, $i>1$, $G(1,k)=1$, $k\geq 0$, $G(i,0)=0$, $i>1$. Note also that $w_i = x_i^{(2)} - x_i$ are the quantities used in [5].

We have

$$P(i,k) = (p^i+q^i)P(i,k-1) + \sum_{u=1}^{i-1}\binom{i}{u}q^u p^{i-u}P(u,k-1), \qquad i\geq 2. \tag{1}$$

Indeed, with probability q^i, nobody advances, we repeat the procedure, with probability p^i, all players are killed, they are resurrected and they try again, with probability $\binom{i}{u}q^u p^{i-u}, 1\leq u\leq i-1$, u players survive.

This equation is also satisfied by $G(i,k)$.

$$x_i = 1 + (p^i+q^i)x_i + \sum_{u=2}^{i-1}\binom{i}{u}q^u p^{i-u}x_u, \qquad i\geq 2, \qquad x_1 = 0, \tag{2}$$

$$x_i^{(2)} = 1 + 2(x_i-1) + (p^i+q^i)x_i^{(2)} + \sum_{u=2}^{i-1}\binom{i}{u}q^u p^{i-u}x_u^{(2)}, \qquad i\geq 2, \qquad x_1^{(2)} = 0. \tag{3}$$

These recursions are self-explanatory and appeared already in the earlier papers.

3. The asymptotic probability distribution of K

We are interested in the asymptotics $f(\eta)$ of $P(n,j)$ and $F(\eta)$ of $G(n,j)$, when n gets large. We use the normalization $\eta := j - \log n^*$. We want to derive these functions in the easiest and shortest way.

We will use two paradigms:

- First, in the urn model, we have asymptotic independence of urns behaviour as far as random variables related to urns with a fixed number of balls are concerned: this is proved in Louchard, Prodinger and Ward [9], with analytic Poissonization-dePoissonization techniques. Also, replacing *binomial* random variables by *Poisson* random variables and using a uniform integrability condition (by computing suitable rates of convergence) are justified, by arguments in [9] and in Louchard and Prodinger [8].

- Next, to compute the asymptotics of the moments, we will use the techniques described in great detail in [7], which are usually simpler than the ones found in the literature. We will encounter extreme-value (Gumbel-like) related distributions functions. The Gumbel distributions function is given by $\exp(-\exp(-x))$. If we compare the approach in this paper with other ones that appeared previously, then we can notice the following. Traditionally, one would stay with exact enumerations as long as possible, and only at a late stage move to asymptotics. Doing this, one would, in terms of asymptotics, carry many unimportant contributions around, which makes the computations quite heavy, especially when it comes to higher moments. Here, however, approximations are carried out as early as possible, and this allows for streamlined (and often automatic) computations of the higher moments.

  Note that the distributions do not converge in the weak sense, they do however converge along subsequences n_m for which the fractional part of $\log n_m$ is constant.

  Thus, the present paper represents another application of the powerful technique that was first presented in [7].

We will consider the maximal non-empty urn. Assume that it contains i balls.

- If $i \geq 2$, this means that we must restart the game with i candidates. Assume that this costs $k \geq 1$ rounds. Denoting by $j-k$ the position of the maximal non-empty urn, we have
$$P_1(n,j) \sim f_1(\eta),$$
where, with $\eta' = j - k - \log n^* = \eta - k$,
$$\begin{aligned} f_1(\eta) &= \sum_{k=1}^{\infty}\sum_{i=2}^{\infty} \Pi\left(-\frac{q}{p}e^{-L\eta'},0\right)\Pi\left(-e^{-L\eta'},i\right)P(i,k) \\ &= \sum_{k=1}^{\infty}\sum_{i=2}^{\infty} \exp\left(-\frac{q}{p}e^{-L\eta'}\right)\exp\left(-e^{-L\eta'}\right)\frac{(e^{-L\eta'})^i}{i!}P(i,k) \\ &= \sum_{k=1}^{\infty}\sum_{i=2}^{\infty} \exp\left(-\frac{1}{p}e^{Lk}e^{-L\eta}\right)\frac{e^{-L\eta i}e^{Lki}}{i!}P(i,k). \end{aligned} \tag{4}$$
Indeed, this corresponds to $i \geq 2$ balls in urn $j-k$ (normalized as η'), no balls to the right of $j-k$ and the end of the game occurs at j (normalized as η). We use asymptotic independence of urns and Poisson limiting random variables.

  All sums should be finite, but, as proved in [8] for similar sums, asymptotically, we can use infinite sums.

  The asymptotic joint distribution of the maximal non-empty urn (η') and its contents (i) was already obtained by Bruss and Grübel in [1] with a different method.
- If $i = 1$, this means that the last alive candidate corresponds to some urn j, which is the last non-empty urn *before* the maximal non-empty urn. The distribution of balls *after* urn j is asymptotically given by $\Pi(\frac{q}{p}e^{-L\eta},u)$. So
$$P_2(n,j) \sim f_2(\eta),$$
where
$$\begin{aligned} f_2(\eta) &= \Pi\left(-\frac{q}{p}e^{-L\eta},1\right)\left(1-\Pi\left(e^{-L\eta},0\right)\right) \\ &= \exp\left(-\frac{q}{p}e^{-L\eta}\right)\frac{q}{p}e^{-L\eta}\left(1-\exp\left(-e^{-L\eta}\right)\right). \end{aligned} \tag{5}$$

Indeed, this corresponds to a non-empty urn at j (normalized as η), one ball to the right of j, and the end of the game occurs at j.

So

$$P(n,j) \sim f(\eta) = f_1(\eta) + f_2(\eta),$$

and $f_i(\eta) > 0$. This is simpler than the expression given in [5]. Note that Knessl [6] also obtains $f(\eta)$ but in a rather complicated way.

The asymptotics of $f_i(\eta)$ are computed as follows. Set $y := e^{-L\eta}$.

- When $\eta \to -\infty$, $y \to \infty$ and we have

$$\begin{aligned}
f_1(\eta) &= \sum_{k=1}^{\infty}\sum_{i=2}^{\infty} \exp\left(-\frac{q}{p}e^{Lk}y\right)\exp\left(-e^{Lk}y\right)\frac{y^i e^{Lki}}{i!}P(i,k) \\
&\leq \sum_{k=1}^{\infty}\sum_{i=2}^{\infty} \exp\left(-\frac{q}{p}e^{Lk}y\right)\exp\left(-e^{Lk}y\right)\frac{y^i e^{Lki}}{i!} \\
&= \sum_{k=1}^{\infty} \exp\left(-\frac{q}{p}e^{Lk}y\right)\left(1-\exp\left(-e^{Lk}y\right)\right) \\
&\sim \int_{u=1}^{\infty} e^{-\frac{q}{p}uy}(1-e^{-uy})\frac{du}{Lu} \\
&= \left[\mathrm{Ei}(1,\frac{q}{p}y) - \mathrm{Ei}(1,\frac{y}{p})\right]/L \\
&\sim \frac{\exp(-\frac{q}{p}y)p}{qLy} \to 0, \qquad y \to \infty.
\end{aligned}$$

$$f_2(\eta) = \exp\left(-\frac{q}{p}y\right)\frac{q}{p}y\left(1-\exp\left(-y\right)\right) \sim \exp\left(-\frac{q}{p}y\right)\frac{q}{p}y \to 0, \qquad y \to \infty.$$

- When $\eta \to \infty$, $y \to 0$, we first find a crude bound $B(k,r_2)$ for $P(i,k)$, k large, $i \leq r_2$. We have

$$B(k,r_2) \leq p^i B(k-1,r_2) + (1-p^i-qp^{i-1})B(k-1,r_2) \leq (1-qp^{r_2})^k \sim C_1^k, \qquad C_1 := e^{-qp^{r_2}} < 1.$$

Set $C_2 := -\ln C_1/L = qp^{r_2}/L > 0$.

We have

$$\begin{aligned}
f_1(\eta) &= \sum_{k=1}^{\infty}\sum_{i=2}^{\infty} \exp\left(-\frac{q}{p}e^{Lk}y\right)\exp\left(-e^{Lk}y\right)\frac{y^i e^{Lki}}{i!}P(i,k) \\
&= V_1 + V_2,
\end{aligned}$$

where, with $r_1 = r_1(y) = -\ln y/(2L)$,

$$\begin{aligned}
V_1 &= \sum_{k=1}^{r_1}\sum_{i=2}^{\infty} \exp\left(-\frac{q}{p}e^{Lk}y\right)\exp\left(-e^{Lk}y\right)\frac{y^i e^{Lki}}{i!}P(i,k) \\
&\leq \sum_{i=2}^{\infty} \frac{y^i e^{Lr_1 i}}{i!} \\
&\leq 1 - e^{y^{1/2}} \to 0, \qquad y \to 0, \\
V_2 &= V_{2,1} + V_{2,2},
\end{aligned}$$

where

$$
\begin{aligned}
V_{2,1} &= \sum_{k=r_1+1}^{\infty} \sum_{i=2}^{r_2-1} \exp\left(-\frac{q}{p}e^{Lk}y\right) \exp\left(-e^{Lk}y\right) \frac{y^i e^{Lki}}{i!} P(i,k) \\
&\leq \sum_{k=r_1+1}^{\infty} \sum_{i=2}^{r_2-1} \exp\left(-\frac{q}{p}e^{Lk}y\right) \exp\left(-e^{Lk}y\right) \frac{y^i e^{Lki}}{i!} C_1^k \\
&\leq \int_{u=y^{-1/2}}^{\infty} e^{-\frac{q}{p}uy}(1-e^{-uy}) \frac{du}{Lu^{1+C_2}} \\
&= \int_{v=y^{1/2}}^{\infty} e^{-\frac{q}{p}v}(1-e^{-v}) y^{C_2} \frac{dv}{Lv^{1+C_2}} \\
&\sim \frac{y^{C_2}}{L(1-C_2)y^{(C_2-1)/2}} = \frac{y^{(C_2+1)/2}}{L(1-C_2)} \to 0, \qquad y \to 0.
\end{aligned}
$$

For further use, we would like to have $r_2 = r_2(y) \to \infty$, $y \to 0$, such that the previous limit remains true. For that, we must have

$$
p^{r_2} \ln y \to -\infty, \qquad r_2 \ln p + \ln(\ln(1/y)) \to \infty, \qquad r_2 \ln(1/p) = o(\ln(\ln(1/y))).
$$

Now we return to our sums

$$
\begin{aligned}
V_{2,2} &= \sum_{k=r_1+1}^{\infty} \sum_{i=r_2}^{\infty} \exp\left(-\frac{q}{p}e^{Lk}y\right) \exp\left(-e^{Lk}y\right) \frac{y^i e^{Lki}}{i!} P(i,k) \\
&\leq \int_{u=y^{-1/2}}^{\infty} e^{-\frac{q}{p}uy} \left(1 - \frac{\Gamma(r_2, uy)}{(r_2-1)!}\right) \frac{du}{Lu} \\
&= \int_{v=y^{1/2}}^{\infty} e^{-\frac{q}{p}v} \int_0^v e^{-t} t^{r_2-1} dt \frac{dv}{(r_2-1)!Lv} \\
&\leq \int_0^{\infty} e^{-\frac{q}{p}v} \int_0^v e^{-t} t^{r_2-1} dt \frac{dv}{(r_2-1)!Lv} = (W_1+W_2)/L,
\end{aligned}
$$

with

$$
\begin{aligned}
W_1 &\leq \int_0^{r_2^{1/2}} \frac{v^{r_2-1}}{r_2(r_2-1)!} dv = \frac{r_2^{r_2/2}}{r_2^2(r_2-1)!} \to 0, \qquad r_2 \to \infty, \\
W_2 &\leq \int_{r_2^{1/2}}^{\infty} e^{-\frac{q}{p}v} \frac{dv}{v} \sim \frac{e^{-\frac{q}{p}r_2^{1/2}}}{r_2^{1/2}} \to 0, \qquad r_2 \to \infty.
\end{aligned}
$$

Next

$$
f_2 = e^{-\frac{q}{p}y} \frac{q}{p} y(1-e^{-y}) \to 0, \qquad y \to 0.
$$

Now we turn to the most interesting distribution, i. e., the probability distribution function $F(\eta)$ such that $G(n,j) = \mathbb{P}(K \leq j) \sim F(\eta)$ with $F(\eta) := \sum_{u=0}^{\infty} f(\eta - u)$. $F(\eta)$ can be computed as follows. With $\eta' = \eta - k$, $\eta'' = \eta - v - u$,

$$
\begin{aligned}
F_1(\eta) &= \sum_{u=0}^{\infty} \sum_{v=1}^{\infty} \sum_{i=2}^{\infty} \exp\left(-\frac{q}{p}e^{-L\eta''}\right) \exp\left(-e^{-L\eta''}\right) \frac{(e^{-L\eta''})^i}{i!} P(i,v) \\
&= \sum_{k=1}^{\infty} \sum_{v=1}^{k} \sum_{i=2}^{\infty} \exp\left(-\frac{q}{p}e^{-L\eta'}\right) \exp\left(-e^{-L\eta'}\right) \frac{(e^{-L\eta'})^i}{i!} P(i,v)
\end{aligned}
$$

$$= \sum_{k=1}^{\infty}\sum_{i=2}^{\infty} \exp\left(-\frac{q}{p}e^{-L\eta'}\right)\exp\left(-e^{-L\eta'}\right)\frac{(e^{-L\eta'})^i}{i!}G(i,k)$$
$$= \sum_{k=1}^{\infty}\sum_{i=2}^{\infty} \exp\left(-\frac{1}{p}e^{-L\eta'}\right)\frac{(e^{-L\eta'})^i}{i!}G(i,k), \tag{6}$$

which is probabilistically obvious. Also

$$F_2(\eta) := \sum_{k=0}^{\infty} f_2(\eta - k),$$

which can also be written as

$$F_2(\eta) = \exp\left(-\frac{q}{p}e^{-L\eta}\right)\frac{q}{p}e^{-L\eta} + \sum_{k=0}^{\infty}\exp\left(-\frac{1}{p}e^{-L\eta'}\right)e^{-L\eta'}, \tag{7}$$

and

$$F(\eta) = F_1(\eta) + F_2(\eta),$$

which leads to the following proposition.

Proposition 3.1.

$$F(\eta) = \exp\left(-\frac{q}{p}e^{-L\eta}\right)\frac{q}{p}e^{-L\eta} + \sum_{k=0}^{\infty}\sum_{i=1}^{\infty}\exp\left(-\frac{1}{p}e^{-L\eta'}\right)\frac{(e^{-L\eta'})^i}{i!}G(i,k). \tag{8}$$

This has also a clear probabilistic interpretation.

$F(\eta)$ is a decent distribution function. Indeed, $\lim_{\eta\to-\infty} F(\eta) = 0$ (proved in the same manner as above) and $\lim_{\eta\to\infty} F(\eta) = 1$, (see (9)) We have observed the following phenomena: $F_1(\eta)$ and $F_2(\eta)$ do have a periodic component on the right. However, $F(\eta)$ does not have any periodicities and moreover it is increasing (as it should).

Indeed, set $y := \exp(-\eta)$. We have

$$F(y) = \exp\left(-\frac{q}{p}y\right)\frac{q}{p}y + \sum_{k=0}^{\infty}\sum_{i=1}^{\infty}\exp\left(-\frac{1}{p}ye^{Lk}\right)\frac{\left(ye^{Lk}\right)^i}{i!}G(i,k).$$

This series is uniformly convergent, so $F(y)$ is continuous. Now we can use the following lemma from Janson [4, Lemma (4.6)]: Assume that Y_n is a sequence of integer-valued random variables that is monotone. If, for every sequence of integers k_n,

$$\mathbb{P}(Y_n \le k_n) = F(k_n - a_n) + o(1), \qquad n \to \infty,$$

for a right-countinuous function F with $\lim_{x\to-\infty} F(x) = 0$, $\lim_{x\to\infty} F(x) = 1$ and a sequence a_n such that $a_n \to \infty$ and $a_{n+1} - a_n \to 0$ as $n \to \infty$, then F is a distribution function.

These conditions are satisfied in our example, with $a_n = \log n$.

Of course, a direct proof would be welcome, but up to now, it did escape us.

4. Asymptotics of the moments of $K - \log n^*$

The asymptotic moments will be computed almost mechanically, thanks to Maple (with some human guidance, as usual). All details about the general machinery can be found in [7]. We will start directly from $f_i(\eta)$. First we have

$$\phi_1(\alpha) = \int_{-\infty}^{\infty} e^{\alpha\eta} f_1(\eta)d\eta = \sum_{k=1}^{\infty}\sum_{i=2}^{\infty}\frac{e^{Lk\widetilde{\alpha}}(1/p)^{-i+\widetilde{\alpha}}}{Li!}\Gamma(i-\widetilde{\alpha})P(i,k),$$

$$\phi_2(\alpha) = \int_{-\infty}^{\infty} e^{\alpha\eta} f_2(\eta) d\eta = \frac{1}{L}\left[\left(\frac{q}{p}\right)^{\widetilde{\alpha}} - q\left(\frac{1}{p}\right)^{\widetilde{\alpha}}\right]\Gamma(1-\widetilde{\alpha}),$$

with $\widetilde{\alpha} := \alpha/L$, $\Re(\alpha) < L$ and $\phi(\alpha) = \phi_1(\alpha) + \phi_2(\alpha)$. We have

$$\lim_{\alpha\to 0} \phi(\alpha) = 1, \tag{9}$$

as it should.

We must now check an "exponential decrease property" for $\phi_\cdot(-Ls)$. ($\phi_\cdot$ stands for ϕ_1 and/or ϕ_2.) We have, with $s = \sigma + \mathbf{i}t$,

$$\begin{aligned} |\phi_1(\alpha)| &\le \sum_{k=1}^{\infty}\sum_{i=2}^{\infty}\left|\frac{e^{-Lks}p^s p^i}{Li!}\Gamma(i+s)\right| \\ &\le \sum_{k=1}^{\infty}\sum_{i=2}^{\infty}\frac{e^{-Lk\sigma}p^\sigma p^i}{Li!}|t|^{i+\sigma-1/2}e^{-\pi|t|/2} \\ &\le \frac{1}{1-e^{-L\sigma}}\frac{p^\sigma e^{p|t|}}{L}|t|^{\sigma-1/2}e^{-\pi|t|/2}. \end{aligned}$$

The check for $\phi_2(\alpha)$ is similar.

The dominant part of the first moment is given by

$$\tilde{m}_1 = \phi'(0) = \tilde{m}_{1,1} + \tilde{m}_{1,2},$$

and, with the notations from the appendix,

$$\begin{aligned} \tilde{m}_{1,1} &= \frac{U_1}{L} - \frac{M}{L}S_1 - \frac{S_2}{L^2}, \\ \tilde{m}_{1,2} &= \frac{p\gamma}{L^2} - \frac{1+pM}{L}, \\ \tilde{m}_1 &= \frac{U_1+\gamma-1}{L} - M - \frac{1}{2}. \end{aligned} \tag{10}$$

To compare with [5], we must replace $\log n^*$ by $\log n + M + 1$ leading to

$$\tilde{m}_1 + M + 1 = \frac{U_1+\gamma-1}{L} + \frac{1}{2}, \tag{11}$$

which is exactly (5) of [5].

The periodic part of the first moment is asymptotically given by

$$w_1 = \sum_{l\neq 0}\varphi_1(\chi_l)e^{-2l\pi\mathbf{i}\log n^*} = \sum_{l\neq 0}\varphi_1(\chi_l)e^{-2l\pi\mathbf{i}\{\log n^*\}}, \tag{12}$$

with

$$\begin{aligned} \chi_l &:= \frac{2l\pi\mathbf{i}}{L}, \\ \varphi_1(\chi_l) &= \varphi_{1,1}(\chi_l) + \varphi_{1,2}(\chi_l), \\ \varphi_{1,i}(\chi_l) &= \phi_i(\alpha)'\big|_{\alpha=-L\chi_l}. \end{aligned}$$

We obtain

$$\varphi_{1,1}(\chi_l) = p^{\chi_l}\left(\frac{T_1(\chi_l)}{L} - M\sum_{i=2}^{\infty}\frac{p^i\Gamma(i+\chi_l)}{Li!} - \sum_{i=2}^{\infty}\frac{p^i\psi(i+\chi_l)\Gamma(i+\chi_l)}{L^2 i!}\right),$$

$$\varphi_{1,2}(\chi_l) = p^{\chi_l}\left(-\frac{p\psi(1+\chi_l)\Gamma(1+\chi_l)}{L^2} - \frac{\Gamma(1+\chi_l)(1+pM)}{L}\right).$$

First we note that replacing in (12) $\log n^*$ by $\log n + M + 1$ allows the elimination of p^{χ_l}. Moreover, grouping some terms together leads to $-R_0(\chi_l)M/L - R_1(\chi_l)/L^2$. This finally gives

$$w_1 = \sum_{l\neq 0}\left[\frac{T_1(\chi_l)}{L} - \frac{\Gamma(\chi_l)(1+\chi_l)}{L}\right]e^{-2l\pi\mathbf{i}\log n}, \tag{13}$$

which again is equivalent to (7) of [5]. This periodic function has 0 mean and small amplitude, for fixed p.

Note that in [6], Knessl obtains a global form for the mean: with $\beta := \{\log n\}$, he gets

$$\begin{aligned}\tilde{m}_1 + w_1 = -\beta + \sum_{j=-\infty}^{\infty}\left[[\![j\geq 0]\!] - \left(1+q^{j-\beta}\right)\exp\left(-q^{j-\beta}\right)\right] \\ + \sum_{j=2}^{\infty}\frac{p^j}{j!}x(j)\sum_{k=-\infty}^{\infty}q^{(k-\beta)j}\exp\left(-q^{k-\beta}\right).\end{aligned}$$

He shows, that, for $\beta = 0$, this is equivalent to $\tilde{m}_1 + w_1$ as given in [5]. We give, in the Appendix, a short complete proof for all β.

Let us now turn to the second moment. We derive

$$\begin{aligned}\tilde{m}_2 &= \phi''(0),\\ \tilde{m}_{2,1} &= \frac{U_2}{L} - 2M\frac{U_1}{L} - 2\frac{UP_1}{L^2} + M^2\frac{S_1}{L} + 2M\frac{S_2}{L^2} + \frac{S_3+S_4}{L^3},\\ \tilde{m}_{2,2} &= \frac{p(\pi^2/6+\gamma^2)}{L^3} - \frac{2\gamma(pM+1)}{L^2} + \frac{pM^2+2M+1}{L}.\end{aligned}$$

Adding the two contributions, this gives

$$\tilde{m}_2 = \frac{-2UP_1+\gamma^2+\pi^2/6-2\gamma}{L^2} + \frac{-\gamma+U_2+1+2M-2M\gamma-2U_1M}{L} + M^2 + M + \frac{1}{3}. \tag{14}$$

We derive the dominant part of the variance of K as

$$\tilde{\sigma}^2 = \tilde{m}_2 - \tilde{m}_1^2 = \frac{-2UP_1+\pi^2/6-1+2(1-\gamma)U_1-U_1^2}{L^2} + \frac{U_1+U_2}{L} + \frac{1}{12},$$

which again is equivalent to (8) of [5]. Note the *ubiquitous* presence of 1/12 in the variance.

The periodic part of the second moment is asymptotically given by (we group terms together in a similar fashion as for w_1)

$$\begin{aligned}\varphi_{2,i}(\chi) &= \phi_i(\alpha)''\big|_{\alpha=-L\chi},\\ w_2 &= \sum_{l\neq 0}\varphi_2(\chi_l)e^{-2l\pi\mathbf{i}\log n^*},\\ w_2 &= \sum_{l\neq 0}\left\{\frac{T_2(\chi_l)}{L} - 2M\frac{T_1(\chi_l)}{L} - 2\frac{T_3(\chi_l)}{L^2} + 2M\frac{R_1(\chi_l)}{L^2} + \frac{R_2(\chi_l)+R_3(\chi_l)}{L^3}\right.\\ &\quad\left. + \Gamma(1+\chi_l)\left[\frac{2\psi(1+\chi_l)}{L^2} + \frac{1}{L} + 2\frac{M}{L}\right]\right\}e^{-2l\pi\mathbf{i}\log n},\end{aligned}$$

which is a new result. Again, this periodic function has 0 mean and small amplitude, for fixed p.

The periodic term of the variance is given (see [7]) by

$$\kappa_2 = w_2 - 2\tilde{m}_1 w_1 - w_1^2.$$

The constant term of w_1^2 is analyzed as follows. (It requires analysis since it does *not* have mean zero, so we like to extract this mean explicitly.)

We can write it as:

$$\frac{1}{L^2}\sum_{l\neq 0} T_1(\chi_l)T_1(-\chi_l) - \frac{2}{L^2}\sum_{l\neq 0} T_1(\chi_l)\Gamma(-\chi_l)(1-\chi_l) + \frac{1}{L^2}\sum_{l\neq 0}\Gamma(\chi_l)(1+\chi_l)\Gamma(-\chi_l)(1-\chi_l).$$

The simplification of quantities like these is described in the survey paper [12]. Thus, we can be brief.

Let us start with the third term. We consider the function

$$F(z) = \frac{L}{Q^z-1}(1+z)(1-z)\Gamma(z)\Gamma(-z),$$

since the term in question will be the sum over some of its residues:

$$\begin{aligned}
I_1 &= \frac{1}{2\pi i}\int_{(1/2)} F(z)dz \\
&= \frac{1}{2\pi i}\int_{(-1/2)} F(z)dz + \sum_{l\neq 0}\Gamma(\chi_l)(1+\chi_l)\Gamma(-\chi_l)(1-\chi_l) + 1 - \frac{L^2}{12} - \frac{\pi^2}{6} \\
&= -\frac{1}{2\pi i}\int_{(-1/2)} \frac{L}{Q^{-z}-1}(1+z)(1-z)\Gamma(z)\Gamma(-z)dz \\
&- \frac{1}{2\pi i}\int_{(-1/2)} L(1+z)(1-z)\Gamma(z)\Gamma(-z)dz \\
&+ \sum_{l\neq 0}\Gamma(\chi_l)(1+\chi_l)\Gamma(-\chi_l)(1-\chi_l) + 1 - \frac{L^2}{12} - \frac{\pi^2}{6}.
\end{aligned}$$

So

$$\begin{aligned}
2I_1 &= -\frac{1}{2\pi i}\int_{(-1/2)} L(1+z)(1-z)\Gamma(z)\Gamma(-z)dz \\
&+ \sum_{l\neq 0}\Gamma(\chi_l)(1+\chi_l)\Gamma(-\chi_l)(1-\chi_l) + 1 - \frac{L^2}{12} - \frac{\pi^2}{6} \\
&= -\sum_{k\geq 1} L(1-k)(1+k)\frac{(-1)^k}{k!}(k-1)! \\
&+ \sum_{l\neq 0}\Gamma(\chi_l)(1+\chi_l)\Gamma(-\chi_l)(1-\chi_l) + 1 - \frac{L^2}{12} - \frac{\pi^2}{6} \\
&= L\sum_{k\geq 1}(k-1)(1+k)\frac{(-1)^k}{k} \\
&+ \sum_{l\neq 0}\Gamma(\chi_l)(1+\chi_l)\Gamma(-\chi_l)(1-\chi_l) + 1 - \frac{L^2}{12} - \frac{\pi^2}{6}
\end{aligned}$$

$$= L(-\frac{1}{4} + \ln 2) + \sum_{l\neq 0} \Gamma(\chi_l)(1+\chi_l)\Gamma(-\chi_l)(1-\chi_l) + 1 - \frac{L^2}{12} - \frac{\pi^2}{6},$$

which leads to

$$I_1 = -\frac{L}{8} + \frac{L\ln 2}{2} + \frac{1}{2}\sum_{l\neq 0} \Gamma(\chi_l)(1+\chi_l)\Gamma(-\chi_l)(1-\chi_l) + \frac{1}{2} - \frac{L^2}{24} - \frac{\pi^2}{12}.$$

But also

$$I_1 = L\sum_{k\geq 1} \frac{(-1)^{k-1}(k+1)(k-1)}{k(Q^k-1)}.$$

Therefore

$$\frac{2}{L}\sum_{k\geq 1} \frac{(-1)^{k-1}(k+1)(k-1)}{k(Q^k-1)} = -\frac{1}{4L} + \frac{\ln 2}{L} + \frac{1}{L^2}\sum_{l\neq 0} \Gamma(\chi_l)(1+\chi_l)\Gamma(-\chi_l)(1-\chi_l) + \frac{1}{L^2} - \frac{1}{12} - \frac{\pi^2}{6L^2},$$

and so

$$\frac{1}{L^2}\sum_{l\neq 0} \Gamma(\chi_l)(1+\chi_l)\Gamma(-\chi_l)(1-\chi_l) = \frac{1}{4L} - \frac{\ln 2}{L} - \frac{1}{L^2} + \frac{1}{12} + \frac{\pi^2}{6L^2} + \frac{2}{L}\sum_{k\geq 1} \frac{(-1)^{k-1}(k+1)(k-1)}{k(Q^k-1)}$$

which is the evaluation of the third term. Now

$$\sum_{l\neq 0} T_1(\chi_l)T_1(-\chi_l) = \sum_{m,n\geq 2} \frac{p^{m+n}}{m!n!} x_m x_n \sum_{l\neq 0} \Gamma(m+\chi_l)\Gamma(n-\chi_l),$$

and

$$\begin{aligned}\sum_{l\neq 0} \Gamma(m+\chi_l)\Gamma(n-\chi_l) &= L\sum_{k\geq 0} \frac{(m+n+k-1)!(-1)^k}{(Q^{m+k}-1)k!}\\ &+ L\sum_{k\geq 0} \frac{(m+n+k-1)!(-1)^k}{(Q^{n+k}-1)k!}\\ &+ L(m+n-1)!2^{-m-n} - (m-1)!(n-1)!.\end{aligned}$$

(The computation is the same as always: take $F(z) = \frac{L}{Q^z-1}\Gamma(m+z)\Gamma(n-z)$ and sum all the residues, which equals zero.)

And we also have terms

$$\sum_{l\neq 0} T_1(\chi_l)(1-\chi_l)\Gamma(-\chi_l) = \sum_{n\geq 2} \frac{p^n}{n!} x_n \sum_{l\neq 0} \Gamma(n+\chi_l)(1-\chi_l)\Gamma(-\chi_l).$$

For this, we can use the formula

$$\begin{aligned}\sum_{l\neq 0} \Gamma(n+\chi_l)(1-\chi_l)\Gamma(-\chi_l) &= L\sum_{k\geq 1} \frac{1}{Q^k-1} \frac{(n+k-1)!(k-1)(-1)^{k-1}}{k!}\\ &+ L\sum_{k\geq 0} \frac{1}{Q^{n+k}-1} \frac{(-1)^k(k+n+1)(k+n-1)!}{k!}\\ &+ L\frac{(n+2)(n-1)!}{2^{n+1}} + (n-1)!\Big[H_{n-1} - 1 - \frac{L}{2}\Big].\end{aligned}$$

So, putting all these various forms together, one would obtain an alternative form of the constant term in w_1^2. Admittedly, it is not too attractive. But it is good to have it, since, as demonstrated in various other papers, to show that certain terms vanish, one needs the alternative forms. Here, however, no cancellations occur.

Note that Knessl [6] also obtains the variance, but again in a global form, without clear separation between dominant and periodic part.

All other moments can be computed in our automatic manner, they are too long to be given here, but Maple knows them (and human guidance could convince it to display it in one way or another).

5. Asymptotics for $p \to 0$, $p \to 1$

We will only deal with first order asymptotics. We first consider ε small and fixed, then we let $n \to \infty$.

5.1. The behaviour for $p \to 0$.

Let

$$\begin{aligned} p &= \varepsilon, \\ q &= 1-\varepsilon, \\ Q &= \frac{1}{1-\varepsilon} \sim 1+\varepsilon, \\ L &= \ln(1-\varepsilon) \sim \varepsilon, \\ n^* &= \frac{n\varepsilon}{1-\varepsilon} \sim n\varepsilon, \\ \ln(x) &= L\log x, \\ M &\sim \ln(\varepsilon)/\varepsilon, \\ \log(n^*) &\sim \frac{1}{\varepsilon}\ln(n\varepsilon) = \frac{1}{\varepsilon}\ln(n) + \frac{\ln(\varepsilon)}{\varepsilon}. \end{aligned}$$

Two variations of η will be used, with indices. For instance η_0 is the original η:

$$\eta_0 \sim j - \frac{1}{\varepsilon}\ln(n) - \frac{\ln(\varepsilon)}{\varepsilon}.$$

We will only give the dominant term of all our expressions and deal with fixed i in x_i, $x_i^{(2)}$.

First, (2), (3) give

$$x_i \sim 1 + (1-i\varepsilon)x_i + i\varepsilon x_{i-1},$$

hence

$$\begin{aligned} x_2 &\sim \frac{1}{2\varepsilon}, \\ x_3 &\sim \frac{5}{6\varepsilon}, \\ x_2^{(2)} &\sim 1 + 2\Big(\frac{1}{2\varepsilon} - 1\Big) + (1-2\varepsilon)x_2^{(2)}, \end{aligned}$$

hence

$$x_2^{(2)} \sim \frac{1}{2\varepsilon^2}.$$

From (1),

$$G(i,k) \sim (1-i\varepsilon)G(i,k-1) + i\varepsilon G(i-1,k-1),$$

hence

$$G(2,k) \sim (1-2\varepsilon)G(2,k-1)+2\varepsilon,$$

and

$$\begin{aligned}G(2,1) &\sim 2\varepsilon,\\ G(2,2) &\sim 2\varepsilon[1+(1-2\varepsilon)],\\ G(2,k) &\sim 2\varepsilon[1+(1-2\varepsilon)+\cdots+(1-2\varepsilon)^{k-1}]\\ &\sim 1-e^{-2k\varepsilon}.\end{aligned}$$

More generally,

$$G(i,k) \sim (1-e^{-k\varepsilon})^i + i(1-e^{-k\varepsilon})^{i-1}e^{-k\varepsilon}. \tag{15}$$

This is probabilistically obvious: as $\varepsilon \to 0$, the geometric distribution converges to an exponential distribution, with distribution function $1-e^{-j\varepsilon}$, and the probability that an urn contains more than 1 ball goes to 0. The distribution function of the next to last full urn (with one ball), i. e., the duration of the game, is asymptotically given by (15). Also, x_i, x_{i-1} can be derived from this distribution function.

Now

$$\begin{aligned}U_1 &\sim \frac{\varepsilon^2}{2}\frac{1}{2\varepsilon} = \frac{\varepsilon}{4},\\ U_2 &\sim \frac{\varepsilon^2}{2}\frac{1}{2\varepsilon^2} = \frac{1}{4},\\ T_1(\chi_l) &\sim \frac{\varepsilon^2}{2}\frac{1}{2\varepsilon}\Gamma\left(2+\chi_l\right) \sim \frac{\varepsilon}{4}\Gamma\left(2+\frac{2l\pi\mathbf{i}}{\varepsilon}\right)\\ &= \frac{\varepsilon}{4}\left(1+\frac{2l\pi\mathbf{i}}{\varepsilon}\right)\frac{2l\pi\mathbf{i}}{\varepsilon}\Gamma\left(\frac{2l\pi\mathbf{i}}{\varepsilon}\right) = \frac{l\pi\mathbf{i}}{2}\left(1+\frac{2l\pi\mathbf{i}}{\varepsilon}\right)\Gamma\left(\frac{2l\pi\mathbf{i}}{\varepsilon}\right),\\ UP_1 &\sim \frac{\varepsilon^2}{2}\frac{1}{2\varepsilon}(1-\gamma) = \frac{\varepsilon}{4}(1-\gamma).\end{aligned}$$

Let us now analyze the first two moments of K. From (10) and (14), with $K_1 := \varepsilon K - \ln(n)$, we derive

$$\begin{aligned}\mathbb{E}(K_1) &\sim \gamma - 1,\\ \mathbb{E}(K_1^2) &\sim \pi^2/6+\gamma^2-2\gamma,\\ \mathbb{V}(K_1) &\sim \pi^2/6-1.\end{aligned}$$

All moments can be similarly obtained, but we give in the sequel a better way to compute them. This leads to

$$\begin{aligned}x_n = \mathbb{E}(K) &= \frac{\mathbb{E}(K_1)+\ln(n)}{\varepsilon},\\ x_n^{(2)} = \mathbb{E}(K^2) &= \frac{\mathbb{E}(K_1^2)+2\ln(n)\mathbb{E}(K_1)+\ln^2(n)}{\varepsilon^2}.\end{aligned}$$

This gives

$$x_n \sim \frac{\gamma-1+\ln(n)}{\varepsilon}$$

and

$$U_1 \sim \sum_{i=2}^{r}\frac{\varepsilon^i x_i}{i} + \sum_{i=r+1}^{\infty}\frac{\varepsilon^i(\gamma-1+\ln(i))}{i\varepsilon}.$$

The second sum is negligible for r sufficiently large.

The periodic component of the mean of K_1 is given, from (13), by

$$w_1 = \sum_{l\neq 0} \frac{\varepsilon}{\varepsilon}\left(\frac{l\pi\mathbf{i}}{2}-1\right)\left(1+\frac{2l\pi\mathbf{i}}{\varepsilon}\right)\Gamma\left(\frac{2l\pi\mathbf{i}}{\varepsilon}\right)e^{-2l\pi\mathbf{i}\ln n/\varepsilon}.$$

This goes to 0 as $\varepsilon \to 0$, due to the rapid decrease of Γ towards $\mathbf{i}\infty$. This is also clear from the K_1 scale, any increase of K by 1 entails an increase of ε for K_1.

Let us now turn to the asymptotic distribution function of K_1. The first part of (8) leads, with $\eta_1 = \varepsilon j - \ln(n)$ to

$$\begin{aligned} D_0(\eta_1) &\sim \exp\left(-\frac{1}{\varepsilon}e^{-\varepsilon(j-\ln(n)/\varepsilon-\ln(\varepsilon)/\varepsilon)}\right)\frac{1}{\varepsilon}e^{-\varepsilon(j-\ln(n)/\varepsilon-\ln(\varepsilon)/\varepsilon)} \\ &\sim \exp(-e^{-\eta_1})e^{-\eta_1}. \end{aligned}$$

The second part of (8) leads asymptotically to

$$\sum_{k=0}^{\infty}\sum_{i=1}^{\infty}\exp(-e^{-\eta_1}e^{k\varepsilon})\frac{(\varepsilon e^{-\eta_1}e^{k\varepsilon})^i}{i!}G(i,k).$$

For $i = 1$, we have $G(i,k) = 1$. This gives, setting $\tau = e^{k\varepsilon}$ and using Euler-McLaurin,

$$D_1(\eta_1) \sim \int_1^\infty \exp(-\tau e^{-\eta_1})e^{-\eta_1}\,d\tau = \exp(-e^{-\eta_1}),$$

which is nothing but the Gumbel distribution function.

For $i \geq 2$, we have, with (15),

$$D_2(\eta_1) \sim \sum_{i=2}^{\infty}\int_1^\infty \exp(-\tau e^{-\eta_1})\frac{(\tau e^{-\eta_1}\varepsilon)^i}{i!}\left[\left(1-\frac{1}{\tau}\right)^i + i\left(1-\frac{1}{\tau}\right)^{i-1}\frac{1}{\tau}\right]\frac{d\tau}{\varepsilon\tau} \sim \mathcal{O}(\varepsilon)$$

uniformly on η_1.

So, finally, we have the following proposition.

Proposition 5.1. *With $\eta_1 = \varepsilon j - \ln(n)$,*

$$F(\eta_1) = \exp(-e^{-\eta_1})e^{-\eta_1} + \exp(-e^{-\eta_1}).$$

This is easily checked to be a decent distribution function. All moments of K_1 are easily derived from this expression, and we recover the first two moments already computed. For instance, denoting by m_i^G the moments of the Gumbel distribution, we have

$$\mathbb{E}(K_1^i) \sim m_i^G - i m_{i-1}^G.$$

Also, all periodic parts go to 0 as $\varepsilon \to 0$.

5.2. **Behaviour for $p \to 1$.** Let

$$\begin{aligned} p &= 1-\varepsilon, \\ q &= \varepsilon, \\ Q &= \frac{1}{\varepsilon}, \\ L &= -\ln(\varepsilon), \\ n^* &= \frac{n(1-\varepsilon)}{\varepsilon}, \end{aligned}$$

$$M = \frac{\ln(1-\varepsilon)}{-\ln(\varepsilon)} \sim \frac{\varepsilon}{\ln(\varepsilon)},$$
$$\log(n^*) \sim \frac{1}{-\ln(\varepsilon)} \ln\Big(\frac{n}{\varepsilon}(1-\varepsilon)\Big) \sim -\frac{\ln(n)}{\ln(\varepsilon)} + 1 + \frac{\varepsilon}{\ln(\varepsilon)},$$
$$\eta_0 \sim j + \frac{\ln(n)}{\ln(\varepsilon)} - 1 - \frac{\varepsilon}{\ln(\varepsilon)}.$$

Again, we deal with fixed i. Equations (2), (3) give

$$x_2 \sim \frac{1}{2\varepsilon} + \mathcal{O}(1),$$
$$x_i \sim 1 + (1-i\varepsilon)x_i + \binom{i}{2}\varepsilon^2 x_2,$$

hence, for fixed i,

$$x_i \sim \frac{1}{i\varepsilon} + \mathcal{O}(1).$$

Also

$$x_i^{(2)} \sim 1 + 2\Big(\frac{1}{i\varepsilon} - 1\Big) + (1-i\varepsilon)x_i^{(2)} + \binom{i}{2}\varepsilon^2 x_2^{(2)},$$

hence,

$$x_i^{(2)} \sim \frac{2}{i^2\varepsilon^2} + \mathcal{O}(\frac{1}{\varepsilon}).$$

Finally,

$$G(i,k) \sim (1-i\varepsilon)G(i,k-1) + i\varepsilon$$

hence,

$$G(i,k) \sim 1 - e^{-ik\varepsilon}.$$

This is probabilistically evident: As $\varepsilon \to 0$, the probability of all candidates being killed at the next round is close to 1, so the game keeps going on, until one of the candidates survives, with probability $i\varepsilon$ (the conditional probability that more than one survives is small). This leads to an exponential distribution, and we recover the dominant term of x_i, $x_i^{(2)}$.

To compute U_1, we need a more precise expression for x_i. After a close examination of (2), we can make the following observations:

- We have
$$x_i \sim \frac{1}{i\varepsilon} + C_0(i) + C_1(i)\varepsilon + C_2(i)\varepsilon^2 + \cdots \tag{16}$$
- C_l (we drop the i-dependency to ease the notation) is a finite sum of decreasing powers of i, starting with degree $l+1$. We denote by $\bar{C}_l$ the coefficient of i^{l+1} in C_l.
- To compute the term of order $1/\varepsilon$ in U_1, we only need $\bar{C}_l$. All other coefficients can be neglected. Indeed
$$\sum_{i=2}^{\infty} \frac{p^i}{i}\bar{C}_0 i \sim \frac{\bar{C}_0}{\varepsilon}, \qquad \sum_{i=2}^{\infty} \frac{p^i}{i} i^u = o(1/\varepsilon), \qquad u < 1.$$
More generally
$$\sum_{i=2}^{\infty} \frac{p^i}{i}\bar{C}_l i^{l+1} \sim \frac{\bar{C}_l l!}{\varepsilon^{l+1}}, \qquad \sum_{i=2}^{\infty} \frac{p^i}{i} i^u = o(1/\varepsilon^{l+1}), \qquad u < l+1.$$

Actually, we will see in the sequel that we must limit ourselves to $i \le 1/\varepsilon$, and

$$\sum_{i=2}^{1/\varepsilon} \frac{p^i}{i} i^{l+1} \sim \int_{2\varepsilon}^{1} e^{-u} \frac{u^l}{\varepsilon^l} \frac{du}{\varepsilon} \sim \frac{I(l)}{\varepsilon^{l+1}}$$

with

$$I(l) := \int_0^1 e^{-u} u^l du < 1,$$

$$I(l) = \sum_0^\infty \frac{(-1)^k}{(k+l+1)k!} = l! - e^{-1} \left[1 + \sum_{i=0}^{l-1} \prod_{k=0}^{i} (l-k) \right]$$

and

$$U_1 \sim \frac{\pi^2 - 1}{6\varepsilon} + \sum_{l=0}^{\infty} \bar{C}_l I(l)/\varepsilon = U_{1,-1}/\varepsilon, \quad \text{say.}$$

- In order to compute $\bar{C}_l$, we can safely replace, in (2),

q^i by 0,

p^i by $\displaystyle\sum_{k=0}^{\infty} \frac{i^k \varepsilon^k (-1)^k}{k!}$,

x_i in the right-hand side by $\displaystyle\frac{1}{i\varepsilon} + \sum_{l=0}^{\infty} \bar{C}_l i^{l+1} \varepsilon^l$,

$$\sum_{u=2}^{i-1} \binom{i}{u} q^u p^{i-u} x_u \ \text{ by } \ \sum_{u=2}^{i-1} \frac{i^u}{u!} \varepsilon^u (1-\varepsilon)^{i-u} \frac{1}{u\varepsilon} = (1-\varepsilon)^i \sum_{u=2}^{i-1} \frac{i^u \varepsilon^u}{u!(1-\varepsilon)^u u\varepsilon}.$$

Now we compute the coefficient of ε^{l+1} in the equation (2), and if we limit ourselves to the highest power of i, this leads to

$$-iC_l + \sum_{v=0}^{l-1} \frac{i^{l+1-v}(-1)^{l+1-v}}{(l+1-v)!} \bar{C}_v i^{v+1} + i^{l+2} \sum_{v=2}^{l+2} \frac{(-1)^{l+2-v}}{(l+2-v)!v!v} = 0.$$

This gives, with $\bar{C}_0 = 1/4$, the following recurrence for $\bar{C}_l$:

$$\bar{C}_l = \sum_{v=0}^{l-1} \frac{(-1)^{l+1-v}}{(l+1-v)!} \bar{C}_v + \sum_{v=2}^{l+2} \frac{(-1)^{l+2-v}}{(l+2-v)!v!v}.$$

It is interesting to note that

$$\sum_{v=2}^{l+2} \frac{(-1)^{l+2-v}}{(l+2-v)!v!v} = \frac{(-1)^l}{(l+2)!} \big(l + 2 - H_{l+2} \big).$$

We further obtain

$$\bar{C}_1 = \frac{-5}{72}, \quad \bar{C}_2 = \frac{1}{288}, \quad \bar{C}_3 = \frac{47}{43200}, \quad \bar{C}_4 = \frac{-7}{86400}, \quad \dots$$

and

$$U_{1,-1} = \frac{\pi^2 - 1}{6} + \frac{1}{4} I(0) + \frac{-5}{72} I(1) + \frac{1}{288} I(2) + \frac{47}{43200} I(3) + \cdots = 0.7852869325\dots$$

The series (16) converges for $i\varepsilon \le 1$ as $\bar{C}_l$ can be shown to be bounded.

Let us now turn to U_2. Proceeding as before, keeping again only the dominant i power, we derive

$$x_i^{(2)} \sim \frac{2}{i^2\varepsilon^2} + \sum_{l=-1}^{\infty} \bar{C}_l^{(2)} \varepsilon^l i^{l+2}$$

with another recurrence for $\bar{C}_l^{(2)}$, i. e.,

$$x_i^{(2)} \sim \frac{2}{i^2\varepsilon^2} + \frac{i}{4\varepsilon} + \frac{-19i^2}{216} + \frac{13i^3\varepsilon}{1728} + \frac{1489i^4\varepsilon^2}{1296000} + \frac{-439i^5\varepsilon^3}{2592000} + \cdots$$

Hence

$$U_2 \sim \frac{U_{2,-2}}{\varepsilon^2},$$

with

$$\begin{aligned} U_{2,-2} &= 2(\zeta(3)-1) + \frac{1}{4}I(0) + \frac{-19}{216}I(1) + \frac{13}{1728}I(2) + \frac{1489}{1296000}I(3) + \frac{-439}{2592000}I(4) + \cdots \\ &= 0.3506617738\ldots \end{aligned}$$

Similarly,

$$UP_1 \sim Up_{1,-1}\frac{\ln(\varepsilon)}{\varepsilon},$$

with

$$Up_{1,-1} = -\sum_{l=0}^{\infty} \bar{C}_l I(l).$$

Setting now $K_1 = K + \frac{\ln(n)}{\ln(\varepsilon)}$ (we keep only the main part of $\log(n^*)$), neglecting unimportant contributions), we get from (10) and (14) the dominant part of the non-periodic component:

$$\begin{aligned} \mathbb{E}(K_1) &\sim -\frac{U_{1,-1}}{\varepsilon\ln(\varepsilon)}, \\ \mathbb{E}(K_1^2) &\sim -\frac{U_{2,-2}}{\varepsilon^2\ln(\varepsilon)}. \end{aligned}$$

More generally

$$\mathbb{E}(K_1^i) \sim -\frac{U_i}{\ln(\varepsilon)}.$$

So it appears as if the non-periodic component of K_1 were asymptotically given by the random variable $K(I)$, where I is distributed with probability distribution

$$\frac{p^i}{iL}. \tag{17}$$

This will be justified later on.

We also get

$$\begin{aligned} x_n = \mathbb{E}(K) &\sim \frac{U_{1,-1}/\varepsilon + \ln(n)}{-\ln(\varepsilon)}, \\ x_n^{(2)} = \mathbb{E}(K^2) &\sim \frac{-\ln(\varepsilon)U_{2,-2}/\varepsilon^2 + 2\ln(n)U_{1,-1}/\varepsilon + \ln^2(n)}{(\ln(\varepsilon))^2}. \end{aligned}$$

We see now why we can limit ourselves to $i \le 1/\varepsilon$ in the computation of U_k. We obtain

$$U_1 \sim \sum_{i=2}^{1/\varepsilon-1} \frac{p^i}{i}x_i + \sum_{i=1/\varepsilon}^{\infty} \frac{p^i}{i}\frac{U_1+\ln(i)}{-\ln(\varepsilon)} \sim \frac{U_{1,-1}}{\varepsilon} + \int_1^{\infty} e^{-u}\frac{U_1+\ln(u)-\ln(\varepsilon)}{-\ln(\varepsilon)}\frac{du}{u}$$

$$\sim \frac{U_{1,-1}}{\varepsilon} + \frac{\delta_1 U_1}{-\ln(\varepsilon)} + \frac{\delta_2}{-\ln(\varepsilon)} + \delta_1,$$

where δ_1 and δ_2 are some constants. So

$$U_1 \sim \frac{U_{1,-1}}{\varepsilon}\Big(1 + \mathcal{O}(1/\ln(\varepsilon))\Big),$$

which justifies our bounds. Similarly for U_2.

The periodic component of the mean of K is given, from (13), by

$$w_1 \sim -\frac{1}{\ln(\varepsilon)} \sum_{l \neq 0} [T_1(\chi_l) - \Gamma(\chi_l)(1+\chi_l)]\, e^{2l\pi\mathbf{i}\ln n/\ln(\varepsilon)}.$$

But $\chi_l \sim -2l\pi\mathbf{i}/\ln(\varepsilon)$ has a small amplitude, so

$$w_1 \sim -\frac{1}{\ln(\varepsilon)} \sum_{l \neq 0} [U_1 - (1+\chi_l)]\, \Gamma(\chi_l) e^{2l\pi\mathbf{i}\ln n/\ln(\varepsilon)} \sim -\frac{1}{\ln(\varepsilon)\varepsilon} U_{1,-1} \sum_{l \neq 0} \Gamma(\chi_l) e^{2l\pi\mathbf{i}\ln n/\ln(\varepsilon)}.$$

The amplitude is of same order as the non-periodic term of K_1, and the period is large (of order $-\ln(\varepsilon)$).

Let us now turn to the asymptotic distribution of K_1. It is easier here to deal with $f(\eta)$. First, we have

$$\eta \sim j + \frac{\ln(n)}{\ln(\varepsilon)}.$$

This is easily explained by the following argument: At each round with n players, $(1-\varepsilon)n$ players are killed, and εn players survive, on average. So, $-\frac{\ln(n)}{\ln(\varepsilon)}$ rounds are necessary, on average, to reach the maximal non-empty urn, with i players, say. These i players need $K(i)$ rounds to end the game. So we must analyze the asymptotic distribution of the number of players in the maximal non-empty urn. Turning first to (5), we see that, when $i = 1$, we have an $\mathcal{O}(1)$ extra cost with respect to $-\frac{\ln(n)}{\ln(\varepsilon)}$.

If $i \geq 2$, we use (4), which shows that the asymptotic probability of a position v of maximal non-empty urn and a number i of players in this urn, is given, with $\eta_1 = v - \log(n^*)$, by

$$\exp\Big(-\frac{1}{p} e^{-L\eta_1}\Big) \frac{e^{-L\eta_1 i}}{i!}.$$

But $L = -\ln(\varepsilon) \gg 1$, so the position v is highly concentrated around $\log(n^*)$, and we can safely use the probability of a number i of balls *independently* of the position v, i. e., the asymptotic distribution

$$\sum_{v=1}^{\infty} \exp\left(-\frac{1}{p} e^{-L\eta_1}\right) \frac{e^{-L\eta_1 i}}{i!}.$$

This is a harmonic sum, and with

$$\phi_3(\alpha) = \int_{-\infty}^{\infty} e^{\alpha\eta_1} \exp\left(-\frac{1}{p} e^{-L\eta_1}\right) \frac{e^{-L\eta_1 i}}{i!} d\eta_1 = \frac{(1/p)^{-i+\tilde{\alpha}}}{Li!} \Gamma(i - \tilde{\alpha}),$$

the asymptotic probability of i balls is given by

$$\phi_3(0) = \frac{p^i}{iL},$$

which is exactly (17). This is an honest probability measure as

$$\sum_{i=1}^{\infty}\frac{p^i}{iL}=1.$$

This leads to the following proposition.

Proposition 5.2. *With $K_1 = K + \frac{\ln(n)}{\ln(\varepsilon)}$, K_1 is asymptotically given by the random variable $K(I)$, where I is distributed with probability distribution $\frac{p^i}{iL}$.*

6. Conclusion

The present paper is another application of the methodology that, whenever the Gumbel distribution is present, moments can be computed in a fairly automatic way.

We hope to extend this series of applications in some future papers.

Appendices.

Appendix A. Some definitions and identities

$$\begin{aligned}
S_1 &:= \sum_{i=2}^{\infty}\frac{p^i}{i} = L-p,\\
S_2 &:= \sum_{i=2}^{\infty}\frac{p^i\psi(i)}{i} = L^2/2-\gamma S_1,\\
S_3 &:= \sum_{i=2}^{\infty}\frac{p^i\psi(i)^2}{i} = -\frac{L^3}{3}+S_5-p\gamma^2,\\
S_4 &:= \sum_{i=2}^{\infty}\frac{p^i\psi(1,i)}{i},\\
S_3+S_4 &= \frac{L^3}{3}-\frac{p\pi^2}{6}-p\gamma^2+\frac{L\pi^2}{6}+L\gamma^2-L^2\gamma,\\
S_5 &:= \sum_{i\ge 1}H_{i-1}^{(2)}\frac{p^i}{i} = L\mathrm{polylog}\left(2,p\right)-L^2\ln\left(p\right)+2\,L\mathrm{polylog}\left(2,q\right)+2\,\mathrm{polylog}\left(3,q\right)-2\,\zeta\left(3\right),\\
R_0(\chi_l) &:= \sum_{i=1}^{\infty}\frac{p^i}{i!}\Gamma(i+\chi_l)=0,\\
R_1(\chi_l) &:= \sum_{i=1}^{\infty}\frac{p^i}{i!}\Gamma(i+\chi_l)\psi(i+\chi_l)=L\Gamma(\chi_l),\\
R_2(\chi_l) &:= \sum_{i=1}^{\infty}\frac{p^i}{i!}\Gamma(i+\chi_l)\psi(1,i+\chi_l),
\end{aligned}$$

$$\begin{aligned}
R_3(\chi_l) &:= \sum_{i=1}^{\infty} \frac{p^i}{i!}\Gamma(i+\chi_l)\psi^2(i+\chi_l),\\
R_2(\chi_l)+R_3(\chi_l) &= \Gamma\left(\chi_l\right)L^2 + 2\Gamma\left(\chi_l\right)L\psi\left(\chi_l\right),\\
T_1(\chi_l) &:= \sum_{i=2}^{\infty} \frac{p^i}{i!}x_i\Gamma(i+\chi_l),\\
T_2(\chi_l) &:= \sum_{i=2}^{\infty} \frac{p^i}{i!}x_i^{(2)}\Gamma(i+\chi_l),\\
T_3(\chi_l) &:= \sum_{i=2}^{\infty} \frac{p^i}{i!}x_i\psi(i+\chi_l)\Gamma(i+\chi_l),\\
U_1 &:= \sum_{i=2}^{\infty} \frac{p^i x_i}{i},\\
U_2 &:= \sum_{i=2}^{\infty} \frac{p^i x_i^{(2)}}{i},\\
U_k &:= \sum_{i=2}^{\infty} \frac{p^i x_i^{(k)}}{i},\\
UP_1 &:= \sum_{i=2}^{\infty} \frac{p^i x_i \psi(i)}{i}.
\end{aligned}$$

Proof We give here proofs of some of these relations.

$$S_1 = \sum_{i\ge 2}\frac{p^i}{i} = -\ln q - p = L - p.$$

$$S_2 = \sum_{i\ge 2}\frac{p^i\psi(i)}{i} = \sum_{i\ge 2}\frac{p^i(H_{i-1}-\gamma)}{i} = \sum_{i\ge 1}\frac{p^i H_{i-1}}{i} - \gamma S_1.$$

Now we know that

$$\frac{1}{1-z}\ln\frac{1}{1-z} = \sum_{n\ge 0} H_n z^n,$$

integrate this between 0 and z, to get

$$\frac{1}{2}\ln^2\frac{1}{1-z} = \sum_{n\ge 0} H_n\frac{z^{n+1}}{n+1};$$

replace z by p, and we get $\frac{1}{2}L^2$.

$$S_3 = \sum_{i\ge 2}\frac{p^i(H_{i-1}-\gamma)^2}{i}$$

$$\begin{aligned}
&= \sum_{i\ge1}\frac{p^iH_{i-1}^2}{i} - 2\gamma\sum_{i\ge1}\frac{p^iH_{i-1}}{i} + \gamma^2\sum_{i\ge2}\frac{p^i}{i}\\
&= \sum_{i\ge1}\frac{p^iH_{i-1}^2}{i} - \gamma L^2 + \gamma^2(L-p)\\
&= \sum_{i\ge1}\frac{p^iH_{i-1}^2}{i} - p\gamma^2.
\end{aligned}$$

Now

$$\frac{1}{1-z}\ln^2\frac{1}{1-z} = \sum_{n\ge0}(H_n^2 - H_n^{(2)})z^n.$$

Integrating again, we get

$$\frac{1}{3}\ln^3\frac{1}{1-z} = \sum_{n\ge0}(H_n^2 - H_n^{(2)})\frac{z^{n+1}}{n+1},$$

from which the desired formula follows by replacing z by p. So

$$-\frac{1}{3}L^3 = \sum_{n\ge1}(H_{n-1}^2 - H_{n-1}^{(2)})\frac{p^n}{n}.$$

We can compute

$$\sum_{i\ge1}H_{i-1}^{(2)}\frac{z^i}{i}$$

as follows:

$$\frac{1}{1-z}\mathrm{dilog}(1-z) = \sum_{i\ge0}H_i^{(2)}z^i.$$

Integration between 0 and p leads to the result:

$$S_5 = L\mathrm{polylog}(2,p) - L^2\ln(p) + 2\,L\mathrm{polylog}(2,q) + 2\,\mathrm{polylog}(3,q) - 2\,\zeta(3).$$

Recall that

$$\mathrm{polylog}(k,p) = \sum_{n\ge1}\frac{p^n}{n^k}.$$

(This formula for S_5 will not be needed.)

$$\begin{aligned}
S_3 + S_4 &= \sum_{i\ge2}\frac{p^i}{i}\Big[\psi^2(i) + \psi(1,i)\Big]\\
&= \sum_{i\ge2}\frac{p^i}{i!}\Gamma''(i)\\
&= \lim_{\varepsilon\to0}\frac{1}{\varepsilon^2}\Bigg[\sum_{i\ge2}\frac{p^i}{i!}\Gamma(i+2\varepsilon) - 2\sum_{i\ge2}\frac{p^i}{i!}\Gamma(i+\varepsilon) + \sum_{i\ge2}\frac{p^i}{i!}\Gamma(i)\Bigg].
\end{aligned}$$

This gives (Maple can compute all the series and perform the limit)

$$\frac{L^3}{3} - \frac{p\pi^2}{6} - p\gamma^2 + \frac{L\pi^2}{6} + L\gamma^2 - L^2\gamma.$$

Now we turn to the quantities involving periodic components. We write χ, but this variable could mean any χ_l:

$$R_0(\chi) = \Gamma(\chi)\sum_{i\ge 1} p^i \binom{i+\chi-1}{i!} = \Gamma(\chi)(q^{-\chi}-1) = 0.$$

Note again that $\Gamma(z)\psi(z) = \Gamma'(z)$. Therefore

$$\begin{aligned} R_1(\chi) &= \sum_{i\ge 1} p^i \frac{\Gamma'(i+\chi)}{i!} \\ &= \lim_{\varepsilon\to 0}\sum_{i\ge 1} p^i \frac{\Gamma(i+\chi+\varepsilon)-\Gamma(i+\chi)}{i!\varepsilon}. \end{aligned}$$

This evaluates as:

$$-\Gamma(\chi)\psi(\chi) - \frac{\Gamma(\chi)\ln(q)}{q^\chi} + \frac{\Gamma(\chi)\psi(\chi)}{q^\chi},$$

which is just $L\Gamma(\chi)$.

Now note that $\Gamma''(z) = \Gamma(z)(\psi^2(z)+\psi(1,z))$. Therefore

$$\begin{aligned} R_2(\chi)+R_3(\chi) &= \sum_{i\ge 1}\frac{p^i}{i!}\Gamma(\chi+i)\Big(\psi^2(\chi+i)+\psi(1,\chi+i)\Big) \\ &= \sum_{i\ge 1}\frac{p^i}{i!}\Gamma''(\chi+i) \\ &= \lim_{\varepsilon\to 0}\sum_{i\ge 1}\frac{p^i}{i!\varepsilon^2}\Big(\Gamma(\chi+i+2\varepsilon)-2\Gamma(\chi+i\varepsilon)+\Gamma(\chi+i)\Big). \end{aligned}$$

This time we get

$$-\Gamma(\chi)\psi(1,\chi) - \Gamma(\chi)(\psi(\chi))^2 + \frac{\Gamma(\chi)(\ln(q))^2}{q^\chi} + \frac{\Gamma(\chi)\psi(1,\chi)}{q^\chi} + \frac{\Gamma(\chi)(\psi(\chi))^2}{q^\chi} - 2\,\frac{\Gamma(\chi)\ln(q)\psi(\chi)}{q^\chi}.$$

Of course, this simplifies, with the final result

$$\Gamma(\chi)L^2 + 2\Gamma(\chi)L\psi(\chi).$$

■

Appendix B. Knessl's equivalence for the mean ($\beta = 0$)

Consider

$$f(z) := \sum_{j\ge 0}[1-(1+q^j)e^{-zq^j}]$$
$$= \sum_{j\ge 0}[1-e^{-zq^j}] - \sum_{j\ge 0} q^j e^{-zq^j}.$$

The first part has Mellin transform

$$\frac{1}{Q^s-1}\Gamma(s).$$

The fundamental strip is $\langle -1, 0\rangle$.

The second part has Mellin transform

$$\frac{1}{Q^{s-1}-1}\Gamma(s).$$

The fundamental strip is $\langle 0, \infty\rangle$.

To make the fundamental strips match, we consider $f_1 = f + \frac{1}{1-q}$. Then

$$f_1^*(s) = \left[\frac{1}{Q^s-1} + \frac{1}{Q^{s-1}-1}\right]\Gamma(s).$$

The fundamental strip is $\langle -1, 0\rangle$.

Consider

$$g(z) := \sum_{j\ge 1}(1+Q^j)e^{-zQ^j}.$$

Its Mellin transform is

$$g^*(s) = \left[\frac{1}{Q^s-1} + \frac{1}{Q^{s-1}-1}\right]\Gamma(s),$$

but this time the fundamental strip is $\langle 1, \infty\rangle$. So,

$$f_1(z) - g(z) = -\sum_{k\in\mathbb{Z}} \mathrm{Residue}_{s=\chi_k}\left[\frac{1}{Q^s-1} + \frac{1}{Q^{s-1}-1}\right]\Gamma(s)z^{-s}$$
$$-\sum_{k\in\mathbb{Z}} \mathrm{Residue}_{s=1+\chi_k}\left[\frac{1}{Q^s-1} + \frac{1}{Q^{s-1}-1}\right]\Gamma(s)z^{-s}.$$

We compute the residues in question:

$$\text{at } s=0: \frac{\ln(z)}{L} + \frac{\gamma}{L} + \frac{1}{2} + \frac{1}{1-q},$$
$$\text{at } s=1: -\frac{1}{Lz},$$
$$\text{at } s=\chi_k: -\frac{1}{L}\Gamma(\chi_k)z^{-\chi_k},$$
$$\text{at } s=1+\chi_k: -\frac{1}{L}\Gamma(1+\chi_k)z^{-1-\chi_k}.$$

Plug in $z = 1$, and we get

$$f_1(1) - g(1) = \frac{\gamma}{L} + \frac{1}{2} + \frac{1}{1-q} - \frac{1}{L} - \frac{1}{L}\sum_{k\ne 0}(1+\chi_k)\Gamma(\chi_k),$$

and therefore

$$f(1)-g(1)=\frac{\gamma}{L}+\frac12-\frac1L-\frac1L\sum_{k\neq 0}(1+\chi_k)\Gamma(\chi_k),$$

which is the formula Knessl derived. Our proof is much simpler.

Appendix C. Knessl's mean equivalence for $\beta \neq 0$

The formula that we are going to derive is[1]

$$\tilde m_1+w_1=-\beta+\sum_{j=-\infty}^{\infty}\Big[[\![j\ge 0]\!]-\Big(1+q^{j-\beta}\Big)\exp\Big(-q^{j-\beta}\Big)\Big]$$
$$+\sum_{j=2}^{\infty}\frac{p^j}{j!}x_j\sum_{k=-\infty}^{\infty}q^{(k-\beta)j}\exp\Big(-q^{k-\beta}\Big).$$

Knessl has not given a formal proof of it, but mentioned that it can be done along the same lines as the formula for $\beta=0$. With our method, it is not hard to write down a complete proof. Note that $\beta=\{\log n\}$. We start with

$$\sum_{k=-\infty}^{\infty}q^{(k-\beta)j}\exp\big(-q^{k-\beta}\big)=q^{-\beta j}\Big(f(q^{-\beta})-g(q^{-\beta})\Big),$$

with

$$f(z):=\sum_{k\ge 0}q^{kj}e^{-q^kz}$$

and

$$g(z):=-\sum_{k<0}q^{kj}e^{-q^kz}=-\sum_{k\ge 1}Q^{kj}e^{-Q^kz}.$$

The Mellin transform of both functions is the same,

$$\frac{1}{1-q^{j-s}}\Gamma(s),$$

but the fundamental strips are $\langle j,\infty\rangle$ for f, and $\langle -\infty,j\rangle$ for g. Therefore

$$f(z)=g(z)-\sum_k \text{Residue}_{s=j+\chi_l}\frac{1}{1-q^{j-s}}\Gamma(s)z^{-s},$$

or

$$f(z)-g(z)=\sum_{l\in\mathbb{Z}}\frac1L\Gamma(j+\chi_l)z^{-j-\chi_l}.$$

Consequently,

$$\begin{aligned}q^{-\beta j}\Big(f(q^{-\beta})-g(q^{-\beta})\Big)&=q^{-\{\log n\}j}\sum_{l\in\mathbb{Z}}\frac1L\Gamma(j+\chi_l)q^{\{\log n\}(j+\chi_l)}\\&=\sum_{l\in\mathbb{Z}}\frac1L\Gamma(j+\chi_l)q^{\{\log n\}\chi_l}\\&=\sum_{l\in\mathbb{Z}}\frac1L\Gamma(j+\chi_l)e^{-2\pi i l\{\log n\}}\end{aligned}$$

[1]Here we use the indicator function ('Iverson's notation') proposed by Knuth et al. [3].

$$= \sum_{l\in\mathbb{Z}} \frac{1}{L}\Gamma(j+\chi_l)e^{-2\pi i l \log n}.$$

Note that the instance $\beta = 0$ of this formula, viz.

$$\sum_{k\in\mathbb{Z}} q^{kj}e^{-q^k} = \frac{1}{L}\sum_{l\in\mathbb{Z}}\Gamma(j+\chi_l),$$

is the formula [6, (A.13)].

Now we consider the remaining part of the quantity under consideration:

$$\begin{aligned}
&\sum_{j=-\infty}^{\infty}\Big[[\![j\ge 0]\!] - \Big(1+q^{j-\beta}\Big)\exp\Big(-q^{j-\beta}\Big)\Big]\\
&= \sum_{j\ge 0}\Big[1-\Big(1+q^{j-\beta}\Big)\exp\Big(-q^{j-\beta}\Big)\Big] - \sum_{j\ge 1}\Big[\Big(1+q^{-j-\beta}\Big)\exp\Big(-q^{-j-\beta}\Big)\Big]\\
&= f(q^{-\beta}) - g(q^{-\beta}),
\end{aligned}$$

with (the auxiliary functions f and g are different from the previous ones)

$$\begin{aligned}
f(z) &= \sum_{j\ge 0}\big[1-\big(1+q^j z\big)\exp\big(-q^j z\big)\big]\\
&= \sum_{j\ge 0}\big[1-\exp\big(-q^j z\big)\big] - \sum_{j\ge 0} q^j z\exp\big(-q^j z\big),\\
g(z) &= \sum_{j\ge 1}\big[\big(1+q^{-j}z\big)\exp\big(-q^{-j}z\big)\big].
\end{aligned}$$

Again, we look at the Mellin transform:

$$f^*(s) = \frac{-1}{1-q^{-s}}\Gamma(s) - \frac{1}{1-q^{-s}}\Gamma(s+1) = -\frac{1}{1-q^{-s}}\Gamma(s)(s+1),$$

and

$$g^*(s) = \frac{q^s}{1-q^s}\Gamma(s) + \frac{q^s}{1-q^s}\Gamma(s+1) = -\frac{1}{1-q^{-s}}\Gamma(s)(s+1) = f^*(s),$$

but in different fundamental strips. Consequently,

$$\begin{aligned}
f(z) &= g(z) + \sum_{l\in\mathbb{Z}}\mathrm{Residue}_{z=\chi_l}\frac{1}{1-q^{-s}}\Gamma(s)(s+1)z^{-s}\\
&= g(z) - \frac{1}{L}\sum_{l\ne 0}\Gamma(\chi_l)(\chi_l+1)z^{-\chi_l} + \log z - \frac{1}{L} + \frac{\gamma}{L} + \frac{1}{2}.
\end{aligned}$$

Therefore

$$\begin{aligned}
&-\beta + \sum_{j=-\infty}^{\infty}\Big[[\![j\ge 0]\!] - \Big(1+q^{j-\beta}\Big)\exp\Big(-q^{j-\beta}\Big)\Big]\\
&= -\beta - \frac{1}{L}\sum_{l\ne 0}\Gamma(\chi_l)(\chi_l+1)q^{\beta\chi_l} + \log(q^{-\beta}) - \frac{1}{L} + \frac{\gamma}{L} + \frac{1}{2}
\end{aligned}$$

$$= -\frac{1}{L}\sum_{l\neq 0}\Gamma(\chi_l)(\chi_l+1)e^{-2\pi i l\{\log n\}} - \frac{1}{L} + \frac{\gamma}{L} + \frac{1}{2}$$
$$= -\frac{1}{L}\sum_{l\neq 0}\Gamma(\chi_l)(\chi_l+1)e^{-2\pi i l\log n} - \frac{1}{L} + \frac{\gamma}{L} + \frac{1}{2}.$$

This ends the proof. It matches with (10) and (13), as it should.

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