Generalized Approximate Counting Revisited

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Abstract

A large class of q-distributions is defined on the stochastic model of Bernoulli trials in which the probability of success (=advancing to the next level) depends geometrically on the number of trials and the level already reached. If the dependency is only on the level already reached, this is an algorithm called *approximate counting.*

Two random variables, X_n (level reached after n trials) and Y_k (number of trials to reach level k) are of interest. We rederive known results and obtain new ones in a consistent way, based on generating functions.

We also discuss asymptotics. The classical instance of *approximate counting* is more interesting from a mathematical point of view. On the other hand, if the number of trials also decreases the probability of success (advancing to the next level), then the limits are constants which are straight-forward to compute.

1 Introduction

The Markov chain

$$\mathbb{P}(X_{n+1} = k+1 \mid X_n = k) = q^{an+bk+c}, \qquad \mathbb{P}(X_{n+1} = k \mid X_n = k) = 1 - q^{an+bk+c} \tag{1}$$

with the initial condition $\mathbb{P}(X_0 = 0) = 1$ was recently revisited by Charalambides [3], also based on some earlier work [2]. We will adopt the notation $p_C(n,k) = \mathbb{P}(X_n = k)$. So, this process starts at time 0 in state 0, and the likelihood to advance to the next state decreases both with time and level already reached.

Sometimes it is more convenient to start in state 1. This amounts to relabel the states from $0, 1, \dots$ to $1, 2, \dots$. Then parameters (a, b, c) must be changed to $(a, b, c-b)$, to have an equivalent model.

Crippa, Simon and Trunz [5] considered a special case, defined by

$$p_{CST}(n,k) = \lambda_{n,k-1} p_{CST}(n-1,k-1) + (1-\lambda_{n,k}) p_{CST}(n-1,k) \tag{2}$$

where $\lambda_{n,k} = q^{a(n-1)+bk}$ and either $(a,b) = (1,0)$ or $(a,b) = (0,1)$. The starting condition is here $p_{CST}(0,1) = 1$.

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This recursion, with $a = 0$, $b = 1$, $c = -1$, i.e., $\lambda_{n,k} = q^{k-1}$, is known as *approximate counting.* This algorithm starts with a counter C initialized to 1. At each increment, we add 1 to C with probability 2^{-C}. After n increments, the counter C contains a good approximation of $\lfloor \log_2 n \rfloor$, with high accuracy and concentration. This was originally analysed by Flajolet [6].

As a motivation, Crippa, Simon, and Trunz mention, apart from approximate counting, an algorithm by Simon to compute the transitive closure on acyclic digraphs. In [4], applications in biology, particle physics and queue-theory are mentioned.

We will reconsider in this paper the recursion (reformulation of the Markov chain (1))

$$p(n,k) = q^{a(n-1)+b(k-1)+c} p(n-1,k-1) + \big(1 - q^{a(n-1)+bk+c}\big) p(n-1,k)$$

with one of the initial conditions $p(0,0) = 1$ or $p(0,1) = 1$, depending on the context.

The aim in this paper is to derive old and new results with a general approach that is based on generating functions. In this way, we will recover as particular cases many results from the literature.

We also discuss asymptotics. This is more interesting for $a = 0$, which is essentially the approximate counting case, with an expectation of order $\log n$. There are several ways to derive these results: Mellin transform (Flajolet [6]), Rice's method (Kirschenhofer and Prodinger [7]), analysis of extreme-value distributions (Louchard and Prodinger [8]), just to name a few. If $a > 0$, then each failed attempt to advance results in an additional punishment, and the expected level that will be reached is just a constant, which is given in a straight-forward way by an infinite series involving the limits of the (explicit forms of the) probabilities. The paper is organized as follows. In Section 2, we collect a few results from q-analysis. Flajolet's explicit formula for approximate counting is analyzed in Section 3. Section 4 considers general formulæ for X_n distribution, with $a, b, c \geq 0$. The moments of X_n are analyzed in Section 5. Section 6 is devoted to the asymptotics of the moments of X_n for $n \to \infty$. Analysis of Y_k (time to reach k) is given in Section 7. Section 8 concludes the paper. Old results will be given as theorems and new results as propositions.

2 Preliminaries

Before we start, we need to collect a few results from q-analysis. They can be found in many textbooks, e.g., [1]. For our probabilistic interpretation, we always assume $0 < q < 1$. We will use the notation

$$(x;q)_n := (1-x)(1-xq)\dots(1-xq^{n-1});$$

for $(q;q)_n$ we sometimes write $(q)_n$ if no misunderstanding is possible. Furthermore we need the Gaussian coefficients

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{(q)_n}{(q)_k (q)_{n-k}};$$

they are polynomials in q and approach the binomial coefficients $\binom{n}{k}$ as $q \to 1$.

We also need $[n]_q := \frac{1-q^n}{1-q}$ and $[n]_q! := [1]_q [2]_q \dots [n]_q = (q)_n/(1-q)^n$.

We recall Euler's two partition identities:

$$\prod_{i=0}^{\infty} (1 - tq^i) = \sum_{n=0}^{\infty} \frac{(-1)^n t^n q^{\binom{n}{2}}}{(q)_n}, \tag{3}$$

$$\prod_{i=1}^{\infty}(1-tq^{i-1})^{-1}=\sum_{n=0}^{\infty}\frac{1}{(q)_n}t^n. \tag{4}$$

and the q-binomial formulæ:

$$\prod_{i=1}^{n}(u+tq^{i-1})=\sum_{k=0}^{n}q^{\binom{k}{2}}\begin{bmatrix}n\\k\end{bmatrix}_q t^k u^{n-k}, \tag{5}$$

$$\frac{1}{(t;q)_n}=\sum_{k=0}^{\infty}\begin{bmatrix}n+k-1\\k\end{bmatrix}_q t^k. \tag{6}$$

We use the (now standard) notation $[z^n]f(z)$ to extract the coefficient of z^n in the series expansion of $f(z)$, as well as Iverson's notation $[\![P]\!]$, which is 1 if P is true, and 0 otherwise.

3 Flajolet's explicit formula

Let us first rederive this formula [6, (46)] in the simplest way: Flajolet proves

Theorem 3.1

$$p(n,k)=\sum_{t=0}^{k-1}\frac{(-1)^t q^{\binom{t}{2}}}{(q)_t(q)_{k-1-t}}(1-q^{k-t})^n.$$

Proof We have

$$p(n,k)=q^{k-1}p(n-1,k-1)+\big(1-q^k\big)p(n-1,k), \quad p(0,1)=1.$$

We will use bivariate *generating functions.*

If we set

$$F(z,u):=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}z^n u^k p(n,k),$$

we derive

$$F(z,u)-u=zuF(z,qu)+zF(z,u)-zF(z,qu),$$

or

$$F(z,u)=\frac{u}{1-z}+\frac{z(u-1)}{1-z}F(z,qu).$$

Iterating, this gives

$$\begin{aligned}F(z,u)&=\frac{u}{1-z}+\frac{z(u-1)}{1-z}\frac{uq}{1-z}+\frac{z(u-1)}{1-z}\frac{z(qu-1)}{(1-z)^2}uq^2\\&\quad+\frac{z(u-1)}{1-z}\frac{z(qu-1)}{1-z}\frac{z(q^2u-1)}{(1-z)^2}uq^3+\cdots\\&=\sum_{j=0}^{\infty}\frac{(-1)^j z^j(u;q)_j uq^j}{(1-z)^{j+1}}.\end{aligned} \tag{7}$$

This expression is already in [5, eq. (17)] but only for the moments. It was independently derived in [9], using a transformation formula due to Heine.

Now we have several ways of computing $[z^n u^k]F(z,u)$.

We write

$$(u;q)_j = \frac{(u;q)_\infty}{(uq^j;q)_\infty}, \tag{8}$$

and with Euler's partition identity, we have

$$\begin{aligned}
p(n,k) &= \sum_{t=0}^{k-1} \frac{(-1)^t q^{\binom{t}{2}}}{(q)_t (q)_{k-1-t}} \sum_{j=0}^{n} (-1)^j q^{(k-1-t)j} q^j [z^{n-j}](1-z)^{-(j+1)} \\
&= \sum_{t=0}^{k-1} \frac{(-1)^t q^{\binom{t}{2}}}{(q)_t (q)_{k-1-t}} (1-q^{k-t})^n, \qquad (9)
\end{aligned}$$

which is exactly Flajolet's formula. ■

There are (at least) 3 other representations, given below by (10), Prop. 3.2, (11).

Second representation.

Letting $a=0$, $c=0$, $b=1$ in the formula [3, (3.2)], we obtain a second expression,

$$p_C(n,k) = \frac{q^{\binom{k}{2}}}{(q)_k} \sum_{j=k}^{n} (-1)^{j-k} \frac{(q)_j}{(q)_{j-k}} \binom{n}{j}, \quad p_C(0,0)=1. \tag{10}$$

This must be equivalent to Flajolet's formula (with n in (9) replaced by $n-1$, as $p_C(0,1) = 1$).

We will give an independent proof of this fact.

$$p(n-1,k) = \sum_{j=0}^{k-1} \frac{(-1)^j q^{\binom{j}{2}}}{(q)_j (q)_{k-1-j}} (1-q^{j-k})^{n-1} = \sum_{j=0}^{k} \frac{(-1)^j q^{\binom{j}{2}}}{(q)_j (q)_{k-j}} (1-q^{k-j})^n.$$

Let us consider the generating function

$$\begin{aligned}
S &= \sum_{k\ge 0} x^k p(n-1,k) \\
&= \sum_{k\ge 0} x^k \sum_{j=0}^{k} \frac{(-1)^j q^{\binom{j}{2}}}{(q)_j (q)_{k-j}} (1-q^{k-j})^n \\
&= \sum_{l=0}^{n} \binom{n}{l} (-1)^l \sum_{k\ge 0} x^k \sum_{j=0}^{k} \frac{(-1)^j q^{\binom{j}{2}}}{(q)_j (q)_{k-j}} q^{(k-j)l} \\
&= \sum_{l=0}^{n} \binom{n}{l} (-1)^l \sum_{j\ge 0} \frac{(-1)^j q^{\binom{j}{2}}}{(q)_j} \sum_{k\ge j} x^k \frac{1}{(q)_{k-j}} q^{(k-j)l} \\
&= \sum_{l=0}^{n} \binom{n}{l} (-1)^l \sum_{j\ge 0} \frac{(-1)^j x^j q^{\binom{j}{2}}}{(q)_j} \sum_{k\ge 0} (xq^l)^k \frac{1}{(q)_k} \\
&= \sum_{l=0}^{n} \binom{n}{l} (-1)^l \frac{1}{(xq^l;q)_\infty} \sum_{j\ge 0} \frac{(-1)^j x^j q^{\binom{j}{2}}}{(q)_j}
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{l=0}^{n} \binom{n}{l} (-1)^l \frac{1}{(xq^l;q)_\infty} (x;q)_\infty \\
&= \sum_{l=0}^{n} \binom{n}{l} (-1)^l (x;q)_l .
\end{aligned}
$$

On the other hand, let us start from the formula [3, eq. (3.2)]

$$
p_C(n,k) = q^{\binom{k}{2}} \sum_{j=k}^{n} (-1)^{j-k} \begin{bmatrix} j \\ k \end{bmatrix}_q \binom{n}{j}
$$

and consider the generating function

$$
\begin{aligned}
T = \sum_{k\geq 0} x^k p_C(n,k) &= \sum_{k\geq 0} x^k q^{\binom{k}{2}} \sum_{j=k}^{n} (-1)^{j-k} \begin{bmatrix} j \\ k \end{bmatrix}_q \binom{n}{j} \\
&= \sum_{j=0}^{n} \binom{n}{j} (-1)^j \sum_{k=0}^{j} x^k q^{\binom{k}{2}} (-1)^k \begin{bmatrix} j \\ k \end{bmatrix}_q \\
&= \sum_{j=0}^{n} \binom{n}{j} (-1)^j \prod_{l=0}^{j-1} (1-q^l x) \\
&= \sum_{j=0}^{n} \binom{n}{j} (-1)^j (x;q)_j .
\end{aligned}
$$

So $S = T$, which ends the proof.

Third representation.

A third expression for Flajolet's formula consists in using a q-binomial in (7) to extract$[u^{k-1}]$. First,

$$
[u^k]F(z,u) = \sum_{j\geq 0} \frac{(-1)^j z^j q^j}{(1-z)^{j+1}} [u^{k-1}](u;q)_j = \sum_{j\geq 0} \frac{(-1)^j z^j q^j}{(1-z)^{j+1}} q^{\binom{k-1}{2}} \begin{bmatrix} j \\ k-1 \end{bmatrix}_q (-1)^{k-1},
$$

and consequently:

Proposition 3.2

$$
p(n,k) = [z^n] \sum_{j\geq 0} \frac{(-1)^j z^j q^j}{(1-z)^{j+1}} q^{\binom{k-1}{2}} \begin{bmatrix} j \\ k-1 \end{bmatrix}_q (-1)^{k-1} = q^{\binom{k-1}{2}} (-1)^{k-1} \sum_{j=k-1}^{n} (-1)^j \begin{bmatrix} j \\ k-1 \end{bmatrix}_q \binom{n}{j} q^j .
$$

Fourth representation.

A fourth expression involving q-Stirling numbers is proved in [5, (14)] by induction. This can be directly done as follows:

First, we compute

$$\begin{aligned}
\sum_n p(n,k)z^n &= q^{\binom{k-1}{2}}(-1)^{k-1}\sum_{j=k-1}^{\infty}(-1)^j z^j \begin{bmatrix} j \\ k-1 \end{bmatrix}_q q^j \frac{1}{(1-z)^{j+1}} \\
&= q^{\binom{k-1}{2}} \frac{q^{k-1}z^{k-1}}{(1-z)^k}\sum_{j=0}^{\infty}\begin{bmatrix} j+k-1 \\ k-1 \end{bmatrix}_q \Big(\frac{-qz}{1-z}\Big)^j \\
&= q^{\binom{k}{2}} \frac{z^{k-1}}{(1-z)^k}\frac{1}{\big(\frac{-qz}{1-z};q\big)_k} \\
&= \frac{q^{\binom{k}{2}}z^{k-1}}{(1-z(1-q))\dots(1-z(1-q^k))}.
\end{aligned}$$

This formula was derived by Flajolet, using a direct combinatorial reasoning.

Now we want to link this to q-Stirling numbers of the second kind (subset Stirling numbers), defined by the recursion

$$\begin{Bmatrix} n \\ k \end{Bmatrix}_q = \begin{Bmatrix} n-1 \\ k-1 \end{Bmatrix}_q + [k]_q \begin{Bmatrix} n-1 \\ k \end{Bmatrix}_q.$$

Let

$$b_k(z) := \sum_n \begin{Bmatrix} n \\ k \end{Bmatrix}_q z^n,$$

then

$$b_k(z) = zb_{k-1}(z) + [k]_q z b_k(z) = \frac{zb_{k-1}(z)}{1-[k]_q z} = \frac{z^k}{(1-[1]_q z)\dots(1-[k]_q z)}.$$

Consequently

$$\begin{Bmatrix} n \\ k \end{Bmatrix}_q = [z^n]\frac{z^k}{(1-[1]_q z)\dots(1-[k]_q z)} = (1-q)^{-n+k}[z^n]\frac{z^k}{(1-z(1-q))\dots(1-z(1-q^k))}.$$

Comparing this with Flajolet's generating function

$$\sum_n p(n,k)z^n = \frac{q^{\binom{k}{2}}z^{k-1}}{(1-z(1-q))\dots(1-z(1-q^k))}$$

we find that

$$p(n,k) = q^{\binom{k}{2}}(1-q)^{n+1-k}\begin{Bmatrix} n+1 \\ k \end{Bmatrix}_q. \tag{11}$$

The moments of (9) will be discussed in Section 5.

4 Analysis of X_n. General formulæ, with $a, b, c \geq 0$.

4.1 General case

Assume as always $0 < q < 1$, which implies $0 \leq q^{an+bk+c} \leq 1$. Again, we will rederive the formula [3, (3.2)] in the simplest way:

Theorem 4.1

$$p_C(n,k) = p(n,k+1) = q^{b\binom{k}{2}} \sum_{j=k}^{n} (-1)^{j-k} q^{a\binom{j}{2}+cj} \begin{bmatrix} n \\ j \end{bmatrix}_{q^a} \begin{bmatrix} j \\ k \end{bmatrix}_{q^b}. \tag{12}$$

We will also obtain another expression:

Proposition 4.2

$$p(n,k) = \sum_{t=0}^{k-1} \frac{(-1)^t q^{b\binom{t}{2}}}{(q^b;q^b)_t (q^b;q^b)_{k-1-t}} (q^{b(k-t)+c}; q^a)_n. \tag{13}$$

Proof of (12). We have

$$p(n,k) = q^{a(n-1)+b(k-1)+c} p(n-1,k-1) + \bigl(1 - q^{a(n-1)+bk+c}\bigr) p(n-1,k), \tag{14}$$

and if we set

$$F(z,u) := \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} z^n u^k p(n,k),$$

we derive

$$F(z,u) - F(z,0) - \sum_{j=1}^{\infty} p(0,j) u^j = q^c z u F(q^a z, q^b u) + z[F(z,u) - F(z,0)] - z q^c [F(q^a z, q^b u) - F(q^a z, 0)],$$

or

$$F(z,u) = \frac{G(z,u)}{1-z} + \frac{z(u-1)q^c}{1-z} F(q^a z, q^b u),$$

with

$$G(z,u) := (1-z)F(z,0) + z q^c F(q^a z, 0) + \sum_{j=1}^{\infty} p(0,j) u^j.$$

Iterating, this gives

$$\begin{aligned} F(z,u) &= \frac{G(z,u)}{1-z} + \frac{z(u-1)q^c}{1-z} \frac{G(q^a z, q^b u)}{1-q^a z} + \frac{z(u-1)q^c}{1-z} \frac{q^a z (q^b u - 1) q^c}{1-q^a z} \frac{G(q^{2a} z, q^{2b} u)}{1-q^{2a} z} \\ &\quad + \frac{z(u-1)q^c}{1-z} \frac{q^a z (q^b u - 1) q^c}{1-q^a z} \frac{q^{2a} z (q^{2b} u - 1) q^c}{1-q^{2a} z} \frac{G(q^{3a} z, q^{3b} u)}{1-q^{3a} z} + \cdots \\ &= \sum_{j=0}^{\infty} \frac{z^j q^{cj} (-1)^j (u;q^b)_j q^{a\binom{j}{2}}}{(z;q^a)_{j+1}} G(z q^{ja}, u q^{jb}). \end{aligned} \tag{15}$$

It is convenient to start with 1 at 0, $p(0,1) = 1$, so $G(z,u) = u$ and $F(z,0) = 0$. Then

$$F(z,u) = \sum_{j=0}^{\infty} \frac{z^j q^{cj} (-1)^j (u;q^b)_j q^{a\binom{j}{2}}}{(z;q^a)_{j+1}} u q^{jb}.$$

Therefore,

$$[u^k] F(z,u) = \sum_{j=0}^{\infty} \frac{z^j q^{cj} (-1)^j q^{jb} q^{a\binom{j}{2}}}{(z;q^a)_{j+1}} [u^{k-1}] (u;q^b)_j$$

$$= \sum_{j=0}^{\infty} \frac{z^j q^{cj} (-1)^j q^{jb} q^{a\binom{j}{2}}}{(z;q^a)_{j+1}} q^{b\binom{k-1}{2}} (-1)^{k-1} \begin{bmatrix} j \\ k-1 \end{bmatrix}_{q^b}$$

and so

$$\begin{aligned} p(n,k) &= [z^n u^k] F(z,u) \\ &= q^{b\binom{k-1}{2}} (-1)^{k-1} \sum_{j=0}^{\infty} q^{cj} (-1)^j q^{jb} q^{a\binom{j}{2}} \begin{bmatrix} j \\ k-1 \end{bmatrix}_{q^b} [z^{n-j}] \frac{1}{(z;q^a)_{j+1}} \\ &= q^{b\binom{k-1}{2}} (-1)^{k-1} \sum_{j=k-1}^{n} (-1)^j \begin{bmatrix} j \\ k-1 \end{bmatrix}_{q^b} \begin{bmatrix} n \\ j \end{bmatrix}_{q^a} q^{a\binom{j}{2}} q^{cj} q^{bj}. \end{aligned} \tag{16}$$

The quantity $p_C(n, k-1)$ from [3] corresponds to $p(n,k)$ as given by (14), with c replaced by $c-b$. Consequently

$$p_C(n,k) = p(n,k+1) = q^{b\binom{k}{2}} \sum_{j=k} (-1)^{j-k} q^{a\binom{j}{2}+cj} \begin{bmatrix} n \\ j \end{bmatrix}_{q^a} \begin{bmatrix} j \\ k \end{bmatrix}_{q^b}.$$

This proves the formula (12) in a simpler way. ■

We can derive a simple new expression from (8) and (15): Here is the proof of (13).
Proof

$$\begin{aligned} p(n,k) &= \sum_{t=0}^{k-1} \frac{(-1)^t q^{b\binom{t}{2}}}{(q^b;q^b)_t (q^b;q^b)_{k-1-t}} \sum_{j=0}^{n} (-1)^j q^{b(k-1-t)j} q^{cj} q^{bj} q^{a\binom{j}{2}} [z^{n-j}] \frac{1}{(z;q^a)_{j+1}} \\ &= \sum_{t=0}^{k-1} \frac{(-1)^t q^{b\binom{t}{2}}}{(q^b;q^b)_t (q^b;q^b)_{k-1-t}} \sum_{j=0}^{n} (-1)^j q^{b(k-t)j} q^{cj} q^{a\binom{j}{2}} \begin{bmatrix} n \\ j \end{bmatrix}_{q^a} \\ &= \sum_{t=0}^{k-1} \frac{(-1)^t q^{b\binom{t}{2}}}{(q^b;q^b)_t (q^b;q^b)_{k-1-t}} \sum_{j=0}^{n} (-1)^j q^{b(k-t)j} q^{cj} [t^j] \prod_{i=1}^{n} (1 + t q^{a(i-1)}) \\ &= \sum_{t=0}^{k-1} \frac{(-1)^t q^{b\binom{t}{2}}}{(q^b;q^b)_t (q^b;q^b)_{k-1-t}} \prod_{i=1}^{n} (1 - q^{b(k-t)} q^c q^{a(i-1)}) \\ &= \sum_{t=0}^{k-1} \frac{(-1)^t q^{b\binom{t}{2}}}{(q^b;q^b)_t (q^b;q^b)_{k-1-t}} (q^{b(k-t)+c}; q^a)_n. \end{aligned}$$

■

Remark. We also obtain $p(n,k)$ from [4, (24)] , with the changes $\alpha = b$, $\beta = a$, $\gamma = c-a$. Crippa and Simon start with 1 at 1, so we must change their n into $n-1$ and our c into $c+a$.

4.2 Case $a = 1,\ b = 0,\ c = 0$.

For that instance, Crippa et al. also establish (by induction) a connection to q-Stirling numbers:

Theorem 4.3

$$p(n,k) = q^{\binom{n}{2}} \left(\frac{1-q}{q}\right)^{n-1+k} s(n,k,1/q).$$

Proof We rederive how this can be done. However, here, we adopt the initial condition $p(0,0) = 1$.

Here is the recursion again for the special case:

$$p(n,k) = q^{n-1} p(n-1,k-1) + \left(1 - q^{n-1}\right) p(n-1,k). \tag{17}$$

Now define

$$a_n(u) = \sum_k p(n,k) u^k,$$

then

$$a_n(u) = u q^{n-1} a_{n-1}(u) + \left(1 - q^{n-1}\right) a_{n-1}(u) = (1 + q^{n-1}(u-1)) a_{n-1}(u)$$

and thus

$$a_n(u) = \prod_{i=0}^{n-1} (1 + q^i (u-1)).$$

Consider q-Stirling numbers recursively defined by

$$s(n,k) = s(n-1,k-1) + [n-1]_q s(n-1,k).$$

Let

$$b_n(u) = \sum_k s(n,k) u^k,$$

then

$$b_n(u) = u b_{n-1}(u) + [n-1]_q b_{n-1}(u) = \prod_{i=1}^{n-1} (u + [i]_q).$$

Therefore (we introduce the q-dependency explicitly)

$$\begin{aligned}
s(n,k,1/q) = [u^k] b_n(u) = [u^k] \prod_{i=1}^{n-1} \left(u + \frac{1 - 1/q^i}{1 - 1/q}\right) &= q^{-\binom{n}{2}} [u^k] \prod_{i=1}^{n-1} \left(u q^i + \frac{q^i - 1}{q - 1} q\right) \\
= q^{-\binom{n}{2}} \left(\frac{q}{1-q}\right)^{n-1} [u^k] \prod_{i=1}^{n-1} \left(u(1-q) q^{i-1} + 1 - q^i\right) \\
= q^{-\binom{n}{2}} \left(\frac{q}{1-q}\right)^{n-1+k} [u^k] \prod_{i=1}^{n-1} \left(u q^i + 1 - q^i\right) &= q^{-\binom{n}{2}} \left(\frac{q}{1-q}\right)^{n-1+k} p(n,k),
\end{aligned}$$

hence

$$p(n,k) = q^{\binom{n}{2}} \left(\frac{1-q}{q}\right)^{n-1+k} s(n,k,1/q).$$

■

Remark. For $a = b$, simplification is possible:

$$\begin{aligned}
p(n, k+1) &= q^{b\binom{k}{2}} \sum_{j=k}^{n} (-1)^{j-k} q^{b\binom{j}{2}+cj} \begin{bmatrix} n \\ j \end{bmatrix}_{q^b} \begin{bmatrix} j \\ k \end{bmatrix}_{q^b} \\
&= q^{b\binom{k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_{q^b} \sum_{j=k}^{n} (-1)^{j-k} q^{b\binom{j}{2}+cj} \begin{bmatrix} n-k \\ j-k \end{bmatrix}_{q^b} \\
&= q^{b\binom{k}{2}+ck} \begin{bmatrix} n \\ k \end{bmatrix}_{q^b} \sum_{j=0}^{n-k} (-1)^{j} q^{b\binom{j+k}{2}+cj} \begin{bmatrix} n-k \\ j \end{bmatrix}_{q^b} \\
&= q^{2b\binom{k}{2}+ck} \begin{bmatrix} n \\ k \end{bmatrix}_{q^b} \sum_{j=0}^{n-k} (-1)^{j} q^{b\binom{j}{2}+bjk+cj} \begin{bmatrix} n-k \\ j \end{bmatrix}_{q^b} \\
&= q^{2b\binom{k}{2}+ck} \begin{bmatrix} n \\ k \end{bmatrix}_{q^b} \prod_{i=k}^{n-1} \left(1 - q^{bi+c}\right) \\
&= q^{2b\binom{k}{2}+ck} \begin{bmatrix} n \\ k \end{bmatrix}_{q^b} (q^{bk+c}; q^b)_{n-k}.
\end{aligned}$$

Specializing further, for $b = 0$ and $c = 1$, this becomes $\binom{n}{k} q^k (1-q)^{n-k}$, which is of course evident.

If we let $b = c = 1$, we get $p(n, k+1) = q^{k^2} \frac{(q)_n^2}{(q)_k^2 (q)_{n-k}}$. Letting n tend to infinity and noticing that the probabilities sum to 1, we get

$$\sum_{k \geq 0} \frac{q^{k^2}}{(q)_k^2} = \frac{1}{(q)_\infty}.$$

This is due to Euler and occurs when enumerating partitions according to Durfee squares [1].

5 The moments

5.1 General case

The moments are derived from $F(z, u)$. We have, starting with 1 at 0 and $b > 0$,

$$\mathbb{E}[X_n^{\underline{i}}] = \sum_{j=i-1}^{n} q^{cj} (-1)^j \begin{bmatrix} n \\ j \end{bmatrix}_{q^a} q^{a\binom{j}{2}} q^{bj} \left. \frac{\partial^i (u(u; q^b)_j)}{\partial u^i} \right|_{u=1}.$$

This leads easily to the following result. Note that $x^{\underline{i}} = x(x-1)\dots(x-i+1)$, and the formula is about the factorial moments:

Proposition 5.1

$$\mathbb{E}[X_n^{\underline{i}}] = i! \sum_{j=i-1}^{n} q^{cj} (-1)^{j-1} \begin{bmatrix} n \\ j \end{bmatrix}_{q^a} q^{a\binom{j}{2}} q^{bj} (q^b; q^b)_{j-1} \times$$

$$\times \left[\sum_{1\le s_1<\cdots<s_{i-2}<j} \frac{q^{bs_1}\dots q^{bs_{i-2}}}{(q^{bs_1}-1)\dots(q^{bs_{i-2}}-1)} \right.$$
$$\left. + \sum_{1\le s_1<\cdots<s_{i-1}<j} \frac{q^{bs_1}\dots q^{bs_{i-1}}}{(q^{bs_1}-1)\dots(q^{bs_{i-1}}-1)} \right]. \tag{18}$$

For instance

$$\mathbb{E}(X_n) = 1 + \sum_{j=1}^{n} q^{cj}(-1)^{j-1} \begin{bmatrix} n \\ j \end{bmatrix}_{q^a} q^{a\binom{j}{2}} q^{bj} (q^b;q^b)_{j-1}, \tag{19}$$

$$\mathbb{E}[X_n^{\underline{2}}] = 2\sum_{j=1}^{n} q^{cj}(-1)^{j-1} \begin{bmatrix} n \\ j \end{bmatrix}_{q^a} q^{a\binom{j}{2}} q^{bj} (q^b;q^b)_{j-1} \left[1 + \sum_{l=1}^{j-1} \frac{q^{bl}}{q^{bl}-1} \right], \tag{20}$$

$$\mathbb{E}[X_n^{\underline{3}}] = 3!\sum_{j=2}^{n} q^{cj}(-1)^{j-1} \begin{bmatrix} n \\ j \end{bmatrix}_{q^a} q^{a\binom{j}{2}} q^{bj} (q^b;q^b)_{j-1} \left[\sum_{l=1}^{j-1} \frac{q^{bl}}{q^{bl}-1} + \sum_{1\le s<l<j} \frac{q^{bl}q^{bs}}{(q^{bl}-1)(q^{bs}-1)} \right].$$

Remark. Using a univariate generating function, Crippa and Simon get the first 2 moments in [4, (27), (28)]. For $a = 0$, the first 2 moments appear in many papers that were written about *approximate counting.*

Remark. Charalambides [3, (3.4) and (3.5)] also computes the first 2 moments, by a lengthy derivation.

5.2 Particular cases

Two particular cases are interesting.

First case: $b = 1$, $a = 0$, $c = 0$.

For $b = 1$, $a = 0$, $c = 0$, we get immediately the moments of *approximate counting.* Since we feel that this result is somehow important, we put it as a proposition:

Proposition 5.2

$$\mathbb{E}[X_n^{\underline{i}}] = i! \sum_{j=i-1}^{n} (-1)^{j-1} \binom{n}{j} q^j (q)_{j-1}$$
$$\times \left[\sum_{1\le s_1<\cdots<s_{i-2}<j} \frac{q^{s_1}\dots q^{s_{i-2}}}{(q^{s_1}-1)\dots(q^{s_{i-2}}-1)} + \sum_{1\le s_1<\cdots<s_{i-1}<j} \frac{q^{s_1}\dots q^{s_{i-1}}}{(q^{s_1}-1)\dots(q^{s_{i-1}}-1)} \right].$$

Remark. [5, Theorem 5] gives the first 2 moments.

Let us briefly review how one can get dominant and periodic parts of the moments of *approximate counting.* As already mentioned, Flajolet derives results for the first 2 moments using the Mellin transform; Kirschenhofer and Prodinger did the same with Rice's method. Using the methods in our recent paper [8, Sections 4.5 and 5.5], all moments can be almost automatically derived. Setting $Q := 1/q$, we obtain the following forms for the mean, the second and third centered moments:

$$\mathbb{E}(X_n - \log_Q n) \sim \tilde{m}_1 + w_1,$$

$$\mathbb{E}(X_n - \log_Q n - \tilde{m}_1 - w_1)^2 \sim \tilde{\mu}_2 + \kappa_2,$$
$$\mathbb{E}(X_n - \log_Q n - \tilde{m}_1 - w_1)^3 \sim \tilde{\mu}_3 + \kappa_3.$$

The detailed expressions are extracted from [8]. Since the formulæ for the third moments are new, we put them as a proposition.

Proposition 5.3 *With the notations*

$$\begin{aligned} L &:= \ln(Q), \\ \chi_l &:= \frac{2\pi \mathbf{i} l}{L}, \\ C_k &:= \sum_{j=1}^{\infty} 1/(Q^j - 1)^k, \end{aligned}$$

we have

$$\begin{aligned}
\tilde{m}_1 &= 1/2 - C_1 + \gamma/L, \\
\tilde{\mu}_2 &= \pi^2/(6L^2) + 1/12 - C_1 - C_2, \\
\tilde{\mu}_3 &= 2\zeta(3)/L^3 - 2C_3 - 3C_2 - C_1, \\
w_1 &= -\sum_{l\neq 0} \Gamma(\chi_l) e^{-2l\pi i \log_Q n}/L, \\
\kappa_2 &= -w_1^2 - 2\gamma w_1/L + 2\sum_{l\neq 0} \Gamma(\chi_l)\psi(\chi_l) e^{-2l\pi i \log_Q n}/L^2, \\
\kappa_3 &= w_1(4L^2 w_1^2 + 12 w_1 L\gamma + 6\gamma^2 - \pi^2)/(2L^2) \\
&\quad - 6(\gamma + w_1 L)\sum_{l\neq 0} \Gamma(\chi_l)\psi(\chi_l) e^{-2l\pi i \log_Q n}/L^3 \\
&\quad - 3\sum_{l\neq 0} \Gamma(\chi_l)\psi^2(\chi_l) e^{-2l\pi i \log_Q n}/L^3 - 3\sum_{l\neq 0} \Gamma(\chi_l)\psi(1,\chi_l) e^{-2l\pi i \log_Q n}/L^3.
\end{aligned}$$

Remark. The (not surprising) fact that $\mathbb{E}[X_n^i] \sim (\log_Q n)^i$ in general, can also be deducted from Rice's method. We do not give a full proof of this but rather sketch a few key steps. It is of course equivalent to consider the factorial moments instead.

$$\begin{aligned}
\mathbb{E}[X_n^{\underline{i}}] &= i! \sum_{j=i-1}^{n} (-1)^{j-1} \binom{n}{j} q^j (q)_{j-1} \\
&\quad \times \left[\sum_{1\le s_1<\cdots<s_{i-2}<j} \frac{q^{s_1}\ldots q^{s_{i-2}}}{(q^{s_1}-1)\ldots(q^{s_{i-2}}-1)} + \sum_{1\le s_1<\cdots<s_{i-1}<j} \frac{q^{s_1}\ldots q^{s_{i-1}}}{(q^{s_1}-1)\ldots(q^{s_{i-1}}-1)} \right].
\end{aligned}$$

Now,

$$\sum_{1\le s_1<\cdots<s_{i-1}<j} \frac{q^{s_1}\ldots q^{s_{i-1}}}{(q^{s_1}-1)\ldots(q^{s_{i-1}}-1)} = \frac{1}{(i-1)!} \sum_{1\le s<j} \left(\frac{q^s}{(q^s-1)}\right)^{i-1} + \text{less important terms}$$

and thus we study

$$i(-1)^{i-1}\sum_{j=i-1}^{n}(-1)^{j-1}\binom{n}{j}q^j(q)_{j-1}\sum_{1\le s<j}\left(\frac{q^s}{1-q^s}\right)^{i-1}.$$

The method (which is described in many textbooks, e. g., [10]) consists in continuing the function

$$q^j(q)_{j-1}\sum_{1\le s<j}\left(\frac{q^s}{1-q^s}\right)^{i-1}$$

to the complex plane, to $\psi(z)$, say, and then computing the residue of

$$i(-1)^{i-1}\frac{\Gamma(n+1)}{\Gamma(n+1-z)\Gamma(-z)}\psi(z)$$

at $s=0$. Observe that

$$\sum_{1\le s<j}\left(\frac{q^s}{1-q^s}\right)^{i-1}=\sum_{s\ge 1}\left(\frac{1}{Q^s-1}\right)^{i-1}-\sum_{s\ge 1}\left(\frac{1}{Q^{s+j-1}-1}\right)^{i-1}$$

so, we use the function

$$\psi(z)=\frac{(q)_\infty}{(q^z)_\infty(Q^z-1)}\left[\sum_{s\ge 1}\left(\frac{1}{Q^s-1}\right)^{i-1}-\sum_{s\ge 1}\left(\frac{1}{Q^{s+z-1}-1}\right)^{i-1}\right].$$

The computation of this residue leads to several terms, since the pole is of order $i+1$. However, the dominant term that comes out is $(\log_Q n)^i$.

Second case: $b=0$.

Another interesting case is $b=0$. If we set $q^b=1-\varepsilon$, this leads to $(q^b;q^b)_j\sim\varepsilon^j j!$ and $\frac{q^{bs_1}}{(q^{bs_1}-1)}\sim\frac{1}{-\varepsilon s_1}$. After a little algebra, we obtain the following result.

Proposition 5.4 *If $b=0$, we have*

$$\mathbb{E}[X_n^{\underline{i}}]=i!q^{c(i-1)}\begin{bmatrix}n\\i-1\end{bmatrix}_{q^a}q^{a\binom{i-1}{2}}.$$

Remarks. Our result include the following special cases:
For $a=1$, $b=0$, $c=0$, [5, (22), (23)], derived there by induction.
The formulæ [4, (25), (26)], with $b=0$ are immediate.
[3, Theorem 3.2] is also immediate.

5.3 q-factorial moments

The q-factorial moments in the general case are given in [3, (3.3)]: The formula is

$$\mathbb{E}\left[(X_n)_{m,q^b}\right]=\frac{1-q^{bm}}{(1-q^b)^m}q^{b\binom{m}{2}}\sum_{j=m}^{n}(-1)^{j-m}q^{a\binom{j}{2}+cj}\begin{bmatrix}n\\j\end{bmatrix}_{q^a}\begin{bmatrix}j\\m\end{bmatrix}_{q^b}(q^b;q^b)_{j-1}.$$

If $a > 0$, this quantity converges to the constant, as $n \to \infty$:

$$\frac{1-q^{bm}}{(1-q^b)^m} q^{b\binom{m}{2}} \sum_{j=m}^{\infty} (-1)^{j-m} q^{a\binom{j}{2}+cj} \frac{1}{(q^a;q^a)_j} \begin{bmatrix} j \\ m \end{bmatrix}_{q^b} (q^b;q^b)_{j-1}.$$

Notice that $\mathbb{E}\left[(X_n)_{m,q^b}\right]$ simplifies for $b = 0$, as in the sum only the term with $j = m$ survives, with the result $m! q^{a\binom{m}{2}+cm} \left[{n \atop m}\right]_{q^a}$. This was derived in [3] in a separate theorem, but is follows readily from the general case. (See the remark above).

6 Asymptotics of the moments of X_n for $n \to \infty$

While the q^b-moments of X_n are quite easy to deal with, as shown above, the proper ones are a bit harder. The results are again constants, but they don't look as pretty as the previous ones.

6.1 General case

We obtain the following result:

Proposition 6.1 *With $a > 0$,*

$$\begin{aligned}\mathbb{E}[X_\infty^{\underline{i}}] = i! \sum_{j=i-1}^{\infty} q^{cj} (-1)^{j-1} \frac{1}{(q^a;q^a)_j} q^{a\binom{j}{2}} q^{bj} (q^b;q^b)_{j-1} \times \\ \times \Bigg[\sum_{s_1=1}^{j-1} \cdots \sum_{s_{i-2}=1}^{j-1} [\![s_1 < s_2 < \ldots < s_{i-2}]\!] \frac{q^{bs_1} \ldots q^{bs_{i-2}}}{(q^{bs_1}-1)\ldots(q^{bs_{i-2}}-1)} \\ + \sum_{s_1=1}^{j-1} \cdots \sum_{s_{i-1}=1}^{j-1} [\![s_1 < s_2 < \ldots < s_{i-1}]\!] \frac{q^{bs_1} \ldots q^{bs_{i-1}}}{(q^{bs_1}-1)\ldots(q^{bs_{i-1}}-1)} \Bigg].\end{aligned}$$

Proof Letting $n \to \infty$, we obtain from (12)

$$p_C(\infty, k) = \frac{q^{(a+b)\binom{k}{2}+ck}}{(q^b;q^b)_k} \sum_{v=0}^{\infty} (-1)^v q^{a[v^2+v(2k-1)]/2+cv} \frac{(q^b;q^b)_{k+v}}{(q^b;q^b)_v (q^a;q^a)_{k+v}}.$$

From (16), we have

$$p(\infty, k+1) = \frac{q^{(a+b)\binom{k}{2}+bk+ck}}{(q^b;q^b)_k} \sum_{v=0}^{\infty} (-1)^v q^{a[v^2+v(2k-1)]/2+bv+cv} \frac{(q^b;q^b)_{k+v}}{(q^b;q^b)_v (q^a;q^a)_{k+v}}.$$

We study the behaviour of the factorial moments:

$$\mathbb{E}\left[X_\infty^{\underline{i}}\right] = \sum_{k=0}^{\infty} k(k-1)\ldots(k-i+1) p(\infty, k). \tag{21}$$

From (18) we derive, with $a > 0$, the proposition. ■

For instance,

$$\mathbb{E}(X_\infty) = 1 + \sum_{j=1}^{\infty} q^{cj}(-1)^{j-1}\frac{1}{(q^a;q^a)_j} q^{a\binom{j}{2}} q^{bj}(q^b;q^b)_{j-1},$$

$$\mathbb{E}[X_\infty^2] = 2\sum_{j=1}^{\infty} q^{cj}(-1)^{j-1}\frac{1}{(q^a;q^a)_j} q^{a\binom{j}{2}} q^{bj}(q^b;q^b)_{j-1}\left[1+\sum_{l=1}^{j-1}\frac{q^{bl}}{q^{bl}-1}\right].$$

Remark. For $b=0$, mean and $E[(X_\infty^2)]$ are derived in [4].

6.2 Case $a = 1$, $b = 0$, $c = 0$.

From (16), we derive in this case ($a=1$, $b=0$, $c=0$) the following result.

Proposition 6.2 *With $a=1$, $b=0$, $c=0$, we have*

$$p_C(\infty,k) = p(\infty,k+1) = q^{\binom{k}{2}}\sum_{v=0}^{\infty}(-1)^v q^{[v^2+v(2k-1)]/2}\frac{\binom{k+v}{v}}{(q)_{k+v}}.$$

The limit when $n\to\infty$ is independent of n and *not* Gaussian (as was suggested in [5]).

In the following, we give an independent proof that $p(\infty,1)=0$ and that the $p(\infty,k)$ sum to 1.

$$p(\infty,1) = \sum_{v=0}^{\infty}(-1)^v q^{\binom{v}{2}}\frac{1}{(q)_v} = (1;q)_\infty = 0,$$

by applying one of Euler's partition identities. And now

$$\begin{aligned}
S &= \sum_{k\ge 0} q^{\binom{k}{2}}\sum_{v=0}^{\infty}(-1)^v q^{[v^2+v(2k-1)]/2}\frac{\binom{k+v}{v}}{(q)_{k+v}} \\
&= \sum_{k,v\ge 0} q^{\binom{k+v}{2}}(-1)^v\frac{\binom{k+v}{v}}{(q)_{k+v}} \\
&= \sum_{n\ge 0}\frac{q^{\binom{n}{2}}}{(q)_n}\sum_{v=0}^{n}(-1)^v\binom{n}{v} \\
&= \sum_{n\ge 0}\frac{q^{\binom{n}{2}}}{(q)_n}[\![n=0]\!] = 1.
\end{aligned}$$

6.3 Other expression

Charalambides [3, (2.11)] expresses the moments in the terms of q-Stirling numbers:

$$\mathbb{E}\left[X_\infty^i\right] = i!\sum_{r=i}^{\infty}(-1)^{r-i}(1-q^b)^{r-i}\frac{(1-q^b)^r}{(q^b;q^b)_r}\mathbb{E}\left[(X_\infty)_{r,q^b}\right] s_{q^b}(r,i).$$

6.4 Tail

When $k \to \infty$, $p(\infty, k)$ leads to the asymptotic equivalent for the tail

$$p_C(\infty, k) \sim \frac{q^{(a+b)\binom{k}{2}+ck}}{(q^a;q^a)_\infty}.$$

If $(a,b) \neq (0,0)$, this can be simplified (with less precision) to

$$p_C(\infty, k) \sim \frac{q^{(a+b)k^2/2}}{(q^a;q^a)_\infty}.$$

Also

$$P_C(\infty, k) := \sum_{i=k}^{\infty} p(\infty, i) \sim \frac{q^{(a+b)\binom{k}{2}+ck}}{(q^a;q^a)_\infty}.$$

These results about tails are new.

7 Analysis of Y_k (time to reach k)

7.1 General case and asymptotics

We obtain the following results:

Proposition 7.1

$$\mathbb{E}([Y_k-1]_{m,q^b}) = \frac{1}{(1-q^b)^m} q^{a(n-1)+b\binom{m+1}{2}} \sum_{m+1 \leq j \leq n} q^{a\binom{j-1}{2}+cj} \begin{bmatrix} n-1 \\ j-1 \end{bmatrix}_{q^a} (-1)^{j-m-1} (q^b;q^b)_{j-1} \begin{bmatrix} j-1 \\ m \end{bmatrix}_{q^b}. \tag{22}$$

Proposition 7.2 *The normalized hitting time Y_k (normalized by $P_C(\infty,k)$) is asymptotically ($k \to \infty$) given by $Y_k = k + U$, where U is a* random variable *with probability generating function*

$$\frac{q^{au}}{(q^a;q^a)_u}(q^a;q^a)_\infty, \tag{23}$$

independent of b.

Proposition 7.3 *The normalized moments are given by*

$$\mathbb{E}(Y_k^i) \sim \sum_{u=0}^{\infty} \frac{q^{au}}{(q^a;q^a)_u}(q^a;q^a)_\infty (k+u)^i = \sum_{v=0}^{i} \sum_{u=0}^{\infty} \frac{q^{au}}{(q^a;q^a)_u}(q^a;q^a)_\infty \binom{i}{v} k^{i-v} u^v, \quad k \to \infty. \tag{24}$$

Proof of (22). The probability $q_C(n,k)$ is given in [3, (4.1), (4.2)] by a rather complicated expression, which can be simplified as

$$q^{a/2(2n-2+k^2)+ck+b\binom{k}{2}} \sum_{v=0}^{n-k} q^{a/2[k(2v-3)+(v-1)(v-2)]+cv} (-1)^v \frac{(q^a;q^a)_{n-1}}{(q^a;q^a)_{n-v-k}(q^a;q^a)_{v+k-1}} \frac{(q^b;q^b)_{v+k-1}}{(q^b;q^b)_v (q^b;q^b)_{k-1}}. \tag{25}$$

Actually, it is simply given, with (12), by

$$q_C(n,k) = p_C(n-1,k-1)q^{a(n-1)+b(k-1)+c}.$$

Indeed, (12) leads to

$$\begin{aligned}
p_C(n,k) &= q^{b\binom{k}{2}}\sum_{j=k}^{n}(-1)^{j-k}q^{a\binom{j}{2}+cj}\begin{bmatrix}n\\ j\end{bmatrix}_{q^a}\begin{bmatrix}j\\ k\end{bmatrix}_{q^b}\\
&= q^{b\binom{k}{2}}\sum_{j=k}^{n}(-1)^{j-k}q^{a\binom{j}{2}+cj}\frac{(q^a;q^a)_n}{(q^a;q^a)_j(q^a;q^a)_{n-j}}\frac{(q^b;q^b)_j}{(q^b;q^b)_k(q^b;q^b)_{j-k}}\\
&= q^{ck}q^{(b+a)\binom{k}{2}}\sum_{v=0}^{n-k}(-1)^v q^{a/2[v^2+v(2k-1)]}q^{cv}\frac{(q^a;q^a)_n}{(q^a;q^a)_{v+k}(q^a;q^a)_{n-v-k}}\frac{(q^b;q^b)_{v+k}}{(q^b;q^b)_k(q^b;q^b)_v}.
\end{aligned}$$

Now we compute a suitable type of moments to get nice results. As was discussed by Charalambides [3], it does not really matter *which* type of moments one computes, as one can always convert.

As already mentioned,

$$q_C(n,k) = q^{a\binom{n}{2}+b\binom{k}{2}+cn-a\binom{n-1}{2}-c(n-1)}\sum_{j=k}^{n}(-1)^{j-k}q^{a\binom{j-1}{2}+c(j-1)}\begin{bmatrix}n-1\\ j-1\end{bmatrix}_{q^a}\begin{bmatrix}j-1\\ k-1\end{bmatrix}_{q^b}$$

or in simplified form:

$$q_C(n,k) = q^{a(n-1)+b\binom{k}{2}}\sum_{j=k}^{n}(-1)^{j-k}q^{a\binom{j-1}{2}+cj}\begin{bmatrix}n-1\\ j-1\end{bmatrix}_{q^a}\begin{bmatrix}j-1\\ k-1\end{bmatrix}_{q^b}.$$

Which type of moments shall we choose in order to get an appealing result?

$$\sum_{1\le j\le n} q^{a(n-1)+a\binom{j-1}{2}+cj}\begin{bmatrix}n-1\\ j-1\end{bmatrix}_{q^a}(-1)^j\sum_{1\le k\le j}(-1)^k q^{b\binom{k}{2}}\begin{bmatrix}j-1\\ k-1\end{bmatrix}_{q^b}\Theta_m(k)$$

$\Theta_m(k)$ must be a suitable "polynomial" of k of degree m, so that we can sum the inner sum in a nice way:

$$S = \sum_{1\le k\le j}(-1)^k q^{b\binom{k}{2}}\begin{bmatrix}j-1\\ k-1\end{bmatrix}_{q^b}\Theta_m(k).$$

We may choose $\Theta_m(k) = (q^b;q^b)_{k-1}/(q^b;q^b)_{k-1-m} = \prod_{i=k-m}^{k-1}(1-q^{bi})$, which is a q^b-factorial. Then,

$$S = \prod_{i=j-m}^{j-1}(1-q^{bi})\sum_{m+1\le k\le j}(-1)^k q^{b\binom{k}{2}}\begin{bmatrix}j-m-1\\ k-m-1\end{bmatrix}_{q^b}.$$

So we are left with

$$S = \frac{(q^b;q^b)_{j-1}}{(q^b;q^b)_{j-m-1}}(-1)^{m+1}\sum_{0\le k\le j-m-1}(-1)^k q^{b\binom{k+m+1}{2}}\begin{bmatrix}j-m-1\\ k\end{bmatrix}_{q^b}$$

$$= \frac{(q^b;q^b)_{j-1}}{(q^b;q^b)_{j-m-1}}(-1)^{m+1}q^{b\binom{m+1}{2}} \sum_{0\le k\le j-m-1} (-1)^k q^{b\binom{k}{2}} \begin{bmatrix} j-m-1 \\ k \end{bmatrix}_{q^b} q^{bk(m+1)}.$$

But we have the formula

$$\sum_{k=0}^{n} q^{b\binom{k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_{q^b} t^k u^{n-k} = \prod_{i=0}^{n-1}(u+tq^{bi}).$$

This applies for $n = j-m-1$, $u = 1$, $t = -q^{b(m+1)}$:

$$\begin{aligned} S &= \frac{(q^b;q^b)_{j-1}}{(q^b;q^b)_{j-m-1}}(-1)^{m+1}q^{b\binom{m+1}{2}} \prod_{i=0}^{j-m-2} (1-q^{b(i+m+1)}) \\ &= \prod_{i=j-m}^{j-1} (1-q^{bi})(-1)^{m+1}q^{b\binom{m+1}{2}} \prod_{i=m+1}^{j-1} (1-q^{bi}) \\ &= (-1)^{m+1}q^{b\binom{m+1}{2}}(q^b;q^b)_{j-1} \begin{bmatrix} j-1 \\ m \end{bmatrix}_{q^b}. \end{aligned}$$

Then the moment becomes

$$q^{a(n-1)+b\binom{m+1}{2}} \sum_{m+1\le j\le n} q^{a\binom{j-1}{2}+cj} \begin{bmatrix} n-1 \\ j-1 \end{bmatrix}_{q^a} (-1)^{j-m-1}(q^b;q^b)_{j-1} \begin{bmatrix} j-1 \\ m \end{bmatrix}_{q^b},$$

which proves (22)

Remark. In the notation of [3], the moment just computed is

$$(1-q^b)^m \sum_k [k-1]_{m,q^b} q_C(n,k).$$

■

If $b = 0$, only one term in the sum survives, and this yields

$$m! q^{a(n-1)+a\binom{m}{2}+cm} \begin{bmatrix} n-1 \\ m \end{bmatrix}_{q^a}.$$

Proof of (23).

When $k \to \infty$, we have the asymptotic equivalent

$$q_C(n,k) \sim q^{a/2(2n-2+k^2)+ck+b\binom{k}{2}} \sum_{v=0}^{n-k} q^{a/2[k(2v-3)+(v-1)(v-2)]+cv} (-1)^v \frac{1}{(q^a;q^a)_{n-v-k}(q^b;q^b)_v},$$

and setting $n = k+u$, this gives

$$q_C(k+u,k) \sim q^{a/2(-k-2+k^2)+ck+b\binom{k}{2}+au} \sum_{v=0}^{u} q^{a/2[2vk+(v-1)(v-2)]+cv} (-1)^v \frac{1}{(q^a;q^a)_{u-v}(q^b;q^b)_v}$$

$$\sim q^{(a+b)\binom{k}{2}+ck+au}\frac{1}{(q^a;q^a)_u}.$$

But we must normalize by $P_C(\infty,k)$; this gives the conditional probability of (23).

Equation (23) is a decent function of u. Indeed by (4), with $t=q^a$,

$$\sum_{u=0}^{\infty}\frac{q^{au}}{(q^a;q^a)_u}=\prod_{i=1}^{\infty}(1-q^aq^{a(i-1)})^{-1}=\prod_{i=1}^{\infty}(1-q^{ai})^{-1}=\frac{1}{(q^a;q^a)_\infty},$$

as it should. ∎

The normalized moments of (24) follow immediately.

Remark.

$$\sum_{v=0}^{i}\sum_{u\geq 0}\frac{q^{au}}{(q^a;q^a)_u}(q^a;q^a)_\infty\binom{i}{v}k^{i-v}u^v=(q^a;q^a)_\infty\sum_{v=0}^{i}\binom{i}{v}k^{i-v}\sum_{u\geq 0}\frac{q^{au}}{(q^a;q^a)_u}u^v.$$

Let us write $r=q^a$. What we need is

$$\sum_{n\geq 0}\frac{t^n}{(r;r)_n}$$

then the inner sum can be obtained via a few differentiations, and $t:=r$. The sum can be written as a product, by Euler's partition identity. However, multiple differentiations lead to iterated sums, and that is all we can do with it.

7.2 Other expression

After normalization, [3, (4.3)] leads, for $k\to\infty$, to

$$\mathbb{E}\left[\frac{(q^a;q^a)_{Y_k+m-1}}{(q^a;q^a)_{Y_k-1}(1-q^a)^m}\right]\sim(1-q^a)^{m+2k}. \tag{26}$$

8 Conclusion

Using generating functions, we have rederived known results and obtained new ones on q-distributions, in an unified and consistent way.

Other forms for the transition probabilities are possible: for instance, in [4], the transition is related to $1-q^{an+bk+c}$ (as opposed to $q^{an+bk+c}$, as in this paper). These generalizations will be the object of future work. It would be interesting to explore how general the recursion might be in order to get some meaningful results.

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