The Swedish Leader Election Protocol: Analysis and Variations*

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1 Introduction

Despite its title, this article does not delve into political issues in Sweden. We analyze a protocol to select one among n players without a centralized agent, a problem that occurs often in distributed computing. The protocol and its variants that we examine here is inspired by the *k-silent elimination* protocol proposed by a team of Swedish researchers a few years ago [1], hence the name.

In general, leader election protocols (see [13, 2, 7, 9, 6, 11] and references therein) have the goal to choose one element (*a player*) out of $n > 0$ given elements, in a distributed decentralized manner. We consider here protocols that start with n *active* players and then proceed in a series of *rounds*. At each round, each player flips a biased coin. With probability q the player tosses heads and passes to the next round. Otherwise, with probability $p = 1 - q$, the player tosses tails and becomes *inactive*, remaining so for the rest of the protocol, at least in principle.

Hence, if W_t denotes the number of active players after round t, we have $W_0 = n$ and for $t > 0$,

$$W_t = \mathrm{Bin}(W_{t-1}, q), \qquad \text{if } W_{t-1} > 0.$$

In all leader election protocols the process will (successfully) finish when at some round we have $W_t = 1$. This single remaining player is then declared the *leader*.

Different leader election protocols result from the way the protocol proceeds in two special situations:

1. Stalled rounds. We say that a round is *stalled* if every active player tosses heads, that is, round t is stalled if $W_t = W_{t-1} > 0$.

2. Null rounds. We say that a round is *null* if every active player tosses tails, that is, round t is null if $W_t = 0$.

In some protocols, the process finishes immediately after a stalled round, declaring every player a "leader". For those protocols, it is of interest to compute the probability that a single leader is elected. In other protocols, stalled rounds constitute no special event, and the rounds simply proceed as usual.

Null rounds pose a different problem to the protocol. In the classical leader election protocol, the players that became inactive in the null round get reactivated for the next round. Thus the number W_t of players still active at the end of round t is now given by

$$W_t = \mathrm{Bin}(W_{t'}, q)$$

where t' is the largest $t' < t$ such that $W_{t'} > 0$. Other protocols will terminate after the first null round or after some number of null rounds have occurred. In those protocols, the probability of successfully declaring a leader (or more) is one of the fundamental

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quantities of interest. Other interesting parameter for these protocols is the number of active players in the last non-null round before the protocol stops (called the *leftovers*).

In this paper, we consider a protocol parameterized by a value $\tau \geq 1$. The protocol is as the classical protocol except that it will stop without declaring a leader if there are τ consecutive null rounds. When $\tau \to \infty$ we have the classical model. On the other hand, when $\tau = 1$ we have the model considered by Kalpathy and Mahmoud [8]. The protocol that we study in this paper is the same as that introduced by Bondesson, Nilsson and Wikstrand [1] under the name k-silent elimination, except that in their proposal the protocol stops at the first stalled round. We call the protocol that we analyze here the *Swedish leader election protocol* in acknowledgement to the work of Bondesson, Nilsson and Wikstrand.

Further variations could also be considered, for instance, by having a rule to stop after τ null rounds (consecutive or not) or to stop after ρ stalled rounds (either consecutive or cumulative), etc.

This paper summarizes some of our preliminary findings; a longer journal version currently under preparation [10] collects these and many more, including variances and distributional results (not just averages, like here) and the analysis of various parameters conditioned to the success or failure of the protocol.

2 A first example: Probability of electing a leader

Let $S_n := S_n(\tau)$ be the probability of success of the protocol. We also use $S_n(t)$ to denote the probability of success when only $t-1$ additional null rounds will be allowed before the protocol fails to elect a leader, for $1 \leq t \leq \tau$. Then

$$(1) \quad S_n(t) = \sum_{1\leq j\leq n} \binom{n}{j} p^{n-j} q^j S_j(\tau) + p^n S_n(t-1), \qquad t > 0, n \geq 2,$$

with $S_n(0) = 0$ for $n \geq 2$, and $S_1(t) = 1$ if $t > 0$. If we define

$$K_n(\tau) = \sum_{1\leq j\leq n} \binom{n}{j} p^{n-j} q^j S_j(\tau)$$

then the recurrence above reads

$$S_n(t) = K_n(\tau) + p^n S_n(t-1), \qquad n \geq 2,$$

which can be easily solved by iteration, so that

$$\begin{aligned} S_n(t) &= K_n(\tau)(1 + p^n + p^{2n} + \cdots + p^{(t-1)n}) \\ &= \frac{1-p^{tn}}{1-p^n} K_n(\tau), \qquad t > 0, n \geq 2. \end{aligned}$$

Therefore we arrive at the following recurrence for $S_n = S_n(\tau)$:

$$(2) \quad S_n = \frac{1-p^{\tau n}}{1-p^n} \sum_{j=1}^{n} \binom{n}{j} p^{n-j} q^j S_j, \qquad n \geq 2,$$

and $S_1 = 1$.

The procedure that follows now is fairly well established and understood. The recurrence for the quantity of interest, say S_n, is translated into a functional equation over the corresponding exponential generating function (EGF). Then Poisson plus Mellin transforms are applied to obtain a solution for the Mellin transform, which is finally inverted via residue computation to obtain precise asymptotic estimates of the original quantity. Equivalently, instead of Mellin transforms, we might use Rice's method. For a general description of these methods and its many applications we refer the reader to the surveys by Flajolet et al. [3, 4], and the books by Szpankowski [14] and by Flajolet and Sedgewick [5]. The procedure sketched above is also known under the name analytic poissonization-depoissonization.

In our particular instance, we start multiplying both sides of (2) by $(1-p^n)$

$$(1-p^n)S_n = (1-p^{\tau n}) \sum_{j=1}^{n} \binom{n}{j} p^{n-j} q^j S_j, \qquad n \geq 2.$$

so that the exponential generating function $S(z) = \sum_{n>0} S_n z^n/n!$ satisfies

$$S(z) - S(pz) = e^{pz} S(qz) - e^{p^{\tau+1}z} S(qp^\tau z) + qp^\tau z.$$

This translates to the following functional equation for the Poisson transform $\hat{S}(z) = e^{-z}S(z)$:

$$\hat{S}(z) - e^{-z}S(pz) = \hat{S}(qz) - e^{-z(1-p^{\tau+1})}S(qp^{\tau}z) + qp^{\tau}ze^{-z}.$$

Rearranging,

$$\hat{S}(z) - \hat{S}(qz) = e^{-z}S(pz) - e^{-z(1-p^{\tau+1})}S(qp^{\tau}z) + qp^{\tau}ze^{-z}$$

The next step is to apply the Mellin transform to both sides of the equation above, with $S^*(s) = \mathcal{M}\left\{\hat{S}(z); s\right\}$, and isolate $S^*(s)$:

$$S^*(s) = \frac{1}{1-q^{-s}}\Big(qp^{\tau}\Gamma(s+1) + \mathcal{M}\left\{e^{-z}S(pz) - e^{-z(1-p^{\tau+1})}S(qp^{\tau}z); s\right\}\Big). \tag{3}$$

The Mellin transform of the second term of the numerator in right hand-side above is

$$\begin{aligned}
&\mathcal{M}\left\{e^{-z}S(pz) - e^{-z(1-p^{\tau+1})}S(qp^{\tau}z); s\right\} \\
&= \int_0^{\infty} \{e^{-z}S(pz) - e^{-z(1-p^{\tau+1})}S(qp^{\tau}z)\}z^{s-1}dz \\
&= \int_0^{\infty} e^{-z}\sum_{k\geq 0} S_k \frac{p^k z^k}{k!} z^{s-1}dz \\
&\quad - \int_0^{\infty} e^{-z(1-p^{\tau+1})}\sum_{k\geq 0} S_k \frac{q^k p^{\tau k}}{k!} z^{s-1}dz \\
&= \sum_{k\geq 0} S_k \frac{p^k\Gamma(s+k)}{k!} \\
&\quad - \sum_{k\geq 0} S_k \frac{q^k p^{\tau k}\Gamma(s+k)}{(1-p^{\tau+1})^{s+k}k!}.
\end{aligned}$$

To obtain the asymptotic behavior S_n as $n \to \infty$, we need to invert the Mellin transform, via

$$\hat{S}(z) = \frac{1}{2\pi\mathbf{i}}\int_{-\frac{1}{2}-\mathbf{i}\infty}^{-\frac{1}{2}+\mathbf{i}\infty} S^*(s)z^{-s}ds.$$

The integral is evaluated by shifting the integration path to the right with negative sign and collecting the residues.

The main contribution to S_n comes from the residue at $s = 0$, namely,

$$\begin{aligned}
C(q,\tau) &:= \mathrm{Res}(S^*(s)z^{-s}; s=0) \\
&= \frac{1}{L}\left(qp^{\tau} + \sum_{k>0}\frac{S_k}{k}\left(p^k - \frac{q^k p^{\tau k}}{(1-p^{\tau+1})^k}\right)\right),
\end{aligned} \tag{4}$$

with $L := \log(1/q)$. The other poles, located at the imaginary axis, are

$$\chi_j = \frac{2\pi\mathbf{i}}{L}j, \qquad j \in \mathbb{Z}\setminus\{0\}$$

whose overall contribution is

$$\frac{1}{L}\sum_{j\neq 0} n^{-\chi_j}\Big(qp^{\tau}\Gamma(\chi_j+1) + \sum_{k>0}\frac{S_k}{k!}\Gamma(\chi_j+k)\big(p^k - \frac{q^k p^{\tau k}}{(1-p^{\tau+1})^{\chi_j+k}}\big)\Big).$$

We have thus the following theorem.

Theorem 1. *Let $S_n(\tau)$ denote the probability that the Swedish leader election protocol succeeds, given n players at the start, and failing if there are τ consecutive null rounds. Then*

$$S_n(\tau) = C(q,\tau) + \delta(\log_Q n) + O(1/n), \qquad \text{as } n \to \infty,$$

where $Q = 1/q$, $L = \log Q$,

$$C(q,\tau) = \frac{1}{L}\left(qp^{\tau} + \sum_{k>0}\frac{S_k}{k}\left(p^k - \frac{q^k p^{\tau k}}{(1-p^{\tau+1})^k}\right)\right),$$

and $\delta(x)$ is a periodic function of "small" amplitude (depending on q and τ) and period 1, namely,

$$\delta(x) = \frac{1}{L}\sum_{j\neq 0} e^{-2x\pi ij}\Big(qp^{\tau}\Gamma(\chi_j+1) + \sum_{k>0}\frac{S_k}{k!}\Gamma(\chi_j+k)\big(p^k - \frac{q^k p^{\tau k}}{(1-p^{\tau+1})^{\chi_j+k}}\big)\Big).$$

An alternative expression for $S_n \sim C(q,t) + \delta(\log_Q n)$ as $n \to \infty$ follows by application of the techniques in [12], with

$$C(q,\tau) = \frac{1}{L}\left(p + \sum_{k>1} S_k \frac{p^k}{k}\frac{1-p^{(\tau-1)k}}{1-p^{\tau k}}\right),$$

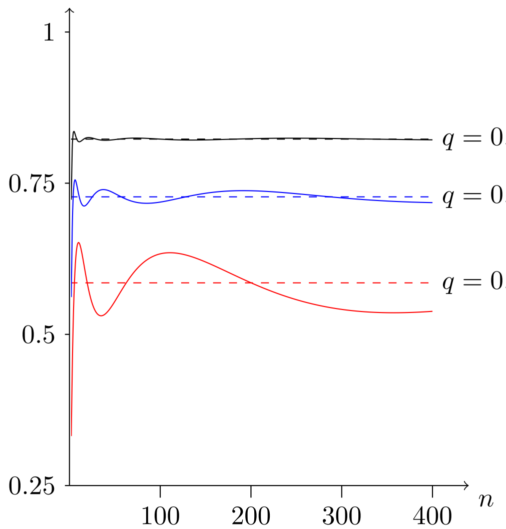

Figure 1: A plot of S_n and C for several values of q; in all plots $\tau = 2$.

$$\delta(x) = \frac{1}{L}\sum_{j\neq 0} e^{-2x\pi \mathbf{i} j}\Big(p\Gamma(\chi_j+1) \\ + \sum_{k>1} S_k \frac{p^k}{k!}\frac{1-p^{(\tau-1)k}}{1-p^{\tau k}}\Gamma(\chi_j+k)\Big).$$

To prove that the expression for S_n above and that in Theorem 1 are equivalent, we use the recursion

$$\frac{1-p^k}{1-p^{\tau k}} S_k = \sum_{j=1}^{k}\binom{k}{j} p^{k-j} q^j S_j$$

in the expressions for $C(q,\tau)$ and $\delta(x)$ above, interchange the order of summation, sum up the inner sum via

$$\sum_{k\geq j} p^{k\tau}\binom{k-1}{j-1} p^{k-j} = \frac{p^{\tau j}}{(1-p^{\tau+1})^j},$$

and simplify.

To conclude this section, we discuss briefly some salient features of S_n by inspection of a few plots—of course, all claims below can be analytically and rigorously proved. Figure 1 shows S_n as a function of n

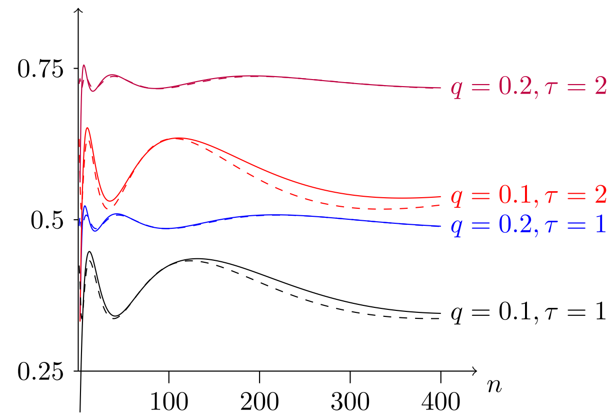

Figure 2: A plot of S_n and $C+\delta(\log_Q n)$ as functions of n.

for several values of q. For each value of q, the figure also shows the corresponding constant C. We have fixed $\tau = 2$ in all cases and we have used $N = 100$ terms of the series to approximate the value of C. The red solid line corresponds to S_n with $q = 0.1$; the red dashed line is $C(0.1, 2)$. Similarly, the blue lines correspond to $q = 0.2$. Finally, the black lines correspond to S_n and C when $q = 0.3$. Observe the periodic fluctuations for all S_n around each corresponding limiting value C; as q increases the fluctuations have smaller amplitude and $C \to 1$.

Figure 2 shows S_n for several values of q and τ (solid lines) and the corresponding approximations as given by Theorem 1 (dashed lines). We have used $N = 20$ terms to approximate C and the inner sum of δ. In turn, only the terms for $j = 1$ and $j = -1$ were used to approximate the value of the fluctuation. The approximation is better as q increases; for values as low as $q = 0.2$ the approximation is already extremely good. And, of course, using more terms to compute $C(q,\tau)$ and $\delta(\log_Q n)$ also improves the approximation.

Figure 3 plots S_n as a function of q for several values of n. The solid blue line corresponds to $n = 10$, the solid red line to $n = 15$ and the solid purple line to $n = 20$. The figure also depicts $C(q,\tau)$ as a function of q (dashed black line). Here, we have used $N = 100$

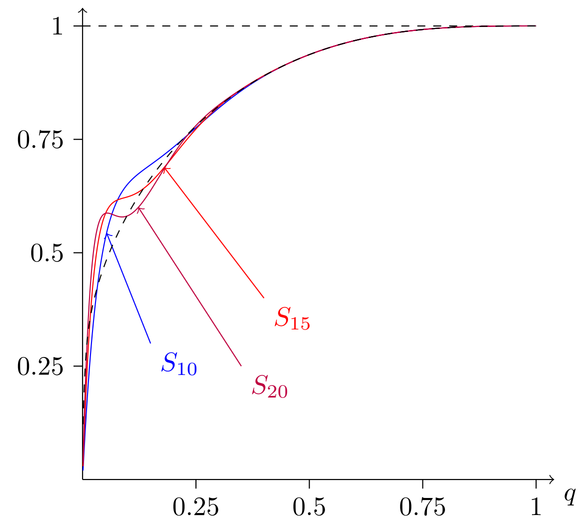

Figure 3: A plot of S_n and C as functions of q, for several values of n; $\tau = 2$ in all cases.

to approximate the value of C. In all cases, we have fixed $\tau = 2$.

The last plot (Figure 4) depicts S_n as a function of τ, for several values of q. Here $n = 20$. The red line is for $q = 0.1$, the blue line for $q = 0.2$, the purple line for $q = 0.3$ and the black line for $q = 0.4$.

3 More parameters

The analysis sketched in the previous section is representative of the procedure and techniques involved in the analysis of various parameters of the protocol.

In general, for many parameters $X_n := X_n(\tau)$ we can set up a recurrence of the form

$$(5)\quad X_n(t) = \sum_{j=1}^{n} \binom{n}{j} p^{n-j} q^j X_j(\tau) + p^n X_n(t-1) + T_n, \qquad t > 0, n \geq 2,$$

for some toll function or sequence $\{T_n\}_{n\geq 0}$. Furthermore, the initial values $X_1 = X_1(\tau)$ and $X_n(0)$, together with the toll function characterize different parameters of the protocol. We assume that $X_0(t) = 0$

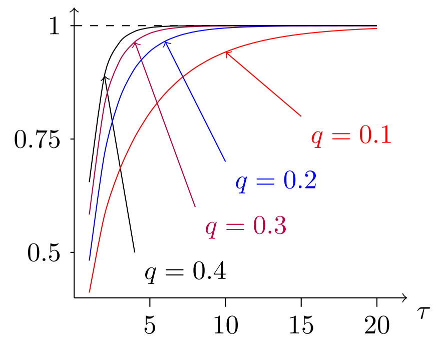

Figure 4: A plot of S_n as a function of τ for several values of q, $n = 20$.

for all $t \geq 0$, and $T_0 = 0$.

Indeed, as we have shown in Section 2, for the probability of success S_n we have $S_1(t) = 1$ for all $t > 0$, $S_n(0) = 0$, and $T_n = 0$ for all $n \geq 2$.

Let us consider other parameters now.

1. For the expected number of rounds R_n, we have that $T_n = 1$ if $n \geq 2$, $R_n = 0$ if $n \leq 1$ (because no more rounds are needed if there is only one or no player remaining) and $R_n(0) = 0$ (because no more rounds are made once the protocol is stopped).

2. For the expected number of null rounds I_n, we have $T_n = p^n$ (with probability p^n the round is null, otherwise it is not) if $n \geq 2$, $T_n = 0$ if $n \leq 1$, $I_1 = 0$ and $I_n(0) = 0$ (since the protocol is stopped).

3. For the expected total number of coins flipped F_n, the toll function is $T_n = n$ for $n \geq 2$, and the initial conditions are $T_n = 0$ if $n \leq 1$, $F_1 = 0$ and $F_n(0) = 0$ for all n.

4. For the expected number L_n of players that were active at the last non-null round (the so-called *left-overs*), we have $L_1 = 0$, $L_n(0) = n$ if $n \geq 2$ and $T_n = 0$ for all n. If we are interested in the expected number L'_n of left-overs *conditioned* on the failure of the protocol, we need only to divide

L_n by the probability of failure $1 - S_n$, as the number of left-overs conditioned on the success of the protocol is 0.

The approach deployed in Section 2 can be applied here to get

$$X^*(s) = \frac{1}{1-q^{-s}}\Bigg(T^*(s) + \Gamma(s+1)(qp^\tau X_1 - (1-p^\tau)T_1)) + \mathcal{M}\Big\{e^{-z}X(pz) - e^{-z(1-p^{\tau+1})}X(qp^\tau z) - e^{-z}T(p^\tau z) + e^{-z}\sum_{n\geq 2} X_n(0)(1-p^n)\frac{(p^\tau z)^n}{n!}; s\Big\}\Bigg), \tag{6}$$

where $X^*(s) = \mathcal{M}\{e^{-z}X(z); s\}$, $T^*(s) = \mathcal{M}\{e^{-z}T(z); s\}$ and

$$X(z) = \sum_{n\geq 1} \frac{X_n}{n!} z^n,$$

$$T(z) = \sum_{n\geq 1} \frac{T_n}{n!} z^n.$$

In particular, the last term of (6) is

$$\begin{aligned}
&\mathcal{M}\Big\{e^{-z}X(pz) - e^{-z(1-p^{\tau+1})}X(qp^\tau z) - e^{-z}T(p^\tau z) + e^{-z}\sum_{k\geq 2} X_k(0)(1-p^k)\frac{(p^\tau z)^k}{k!}; s\Big\} \\
&= \sum_{k\geq 1}\frac{X_k}{k!}\Gamma(s+k)\Big(p^k - \frac{q^k p^{\tau k}}{(1-p^{\tau+1})^{s+k}}\Big) - \sum_{k\geq 1}\frac{T_k}{k!}p^{\tau k}\Gamma(s+k) + \sum_{k\geq 2}\frac{X_k(0)}{k!}p^{\tau k}(1-p^k)\Gamma(s+k).
\end{aligned} \tag{7}$$

The Mellin transform given by (6) will be defined in some strip of the complex plane, for instance, $\Re s > -1$, and will have poles at $s = 0$ and $s = \chi_j$, for all $j \in \mathbb{Z} \setminus \{0\}$, where

$$\chi_j = \frac{2\pi \mathbf{i}}{L} j,$$

as well as possibly other poles. The location of the other poles and the fundamental strip will ultimately depend on the (rate of growth of) sequence $\{T_n\}_{n\geq 0}$. So, in general, we will have that $X_n \sim X_{n,1} + X_{n,2}$, where $X_{n,2}$ is the contribution coming from (7) at the poles $s = 0$ and $s = \chi_j$:

$$\begin{aligned}
X_{n,2} &= \frac{1}{L}\sum_{k\geq 1}\frac{X_k}{k}\left(p^k - \frac{q^k p^{\tau k}}{(1-p^{\tau+1})^k}\right) \\
&\quad + \frac{1}{L}\sum_{k\geq 2}\frac{X_k(0)}{k}p^{\tau k}(1-p^k) - \frac{1}{L}\sum_{k\geq 1}\frac{T_k}{k}p^{\tau k} \\
&\quad + \delta_{X,2}(\log_Q n),
\end{aligned} \tag{8}$$

$$\begin{aligned}
\delta_{X,2}(x) = \frac{1}{L}\sum_{j\neq 0} e^{-2\pi \mathbf{i} jx}\Bigg\{&\sum_{k\geq 1}\frac{X_k}{k!}\Big(p^k - \frac{q^k p^{\tau k}}{(1-p^{\tau+1})^{k+\chi_j}}\Big)\Gamma(\chi_j + k) \\
&+ \sum_{k\geq 2}\frac{X_k(0)}{k!}(1-p^k)p^{\tau k}\Gamma(\chi_j + k) \\
&- \sum_{k\geq 1}\frac{T_k}{k!}p^{\tau k}\Gamma(\chi_j + k)\Bigg\}
\end{aligned} \tag{9}$$

We will find convenient to introduce the following notation, since it occurs often in the examples that we will consider later: given any sequence $A = \{A_n\}_{n\geq 1}$,

$$C(A; q, \tau) := \frac{1}{L}\sum_{k\geq 1}\frac{A_k}{k}\left(p^k - \frac{q^k p^{\tau k}}{(1-p^{\tau+1})^k}\right).$$

Notice that as long as A_n has polynomially bounded growth, the series $C(A; q, \tau)$ converges to a constant that depends on q and τ.

Let's now move on to the concrete examples. The expected number of rounds R_n before a leader is elected or the protocol fails fits into the general framework we are discussing in this section. Indeed, we need just to specialize (6) for $X_n \equiv R_n$, with $T_n = 1$ for $n > 1$, $T_1 = 0$, $X_1 \equiv R_1 = 0$ and $X_n(0) \equiv R_n(0) = 0$. Then $T(z) = e^z - z - 1$ and

$$T^*(s) = -\Gamma(s+2)/s,$$

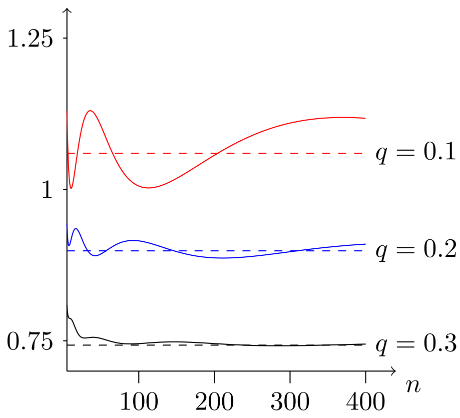

Figure 5: A plot of $R_n - \log_Q n$ and the approximation given by Eq. (10), for several values of q; in all plots $\tau = 2$.

$$R^*(s) = \frac{1}{1-q^{-s}}\Big(-s^{-1}\Gamma(s+2) + \mathcal{M}\{\ldots; s\}\Big),$$

in the fundamental strip $-2 < \Re s < -1$ (see [3]); we have refrained from writing here the second term in full to avoid cluttering.

Again, the asymptotic behavior of R_n for large n can be found by inversion of the Mellin transform $R^*(s)$. The main contribution comes from the pole at $s = 0$, and the poles at $s = \chi_j$ contribute a fluctuating term $\delta_R(\log_Q n)$ which is $O(1)$:

$$R_n = \log_Q n + \frac{\gamma}{L} + \frac{1}{2} - \frac{1}{L} + \frac{1}{L}\big(p^\tau + \log(1-p^\tau)\big) + C(R; q, \tau) + \delta_R(\log_Q n) + O(n^{-1}\log n).$$

Figure 5 compares the value of $R_n - \log_Q n$ to the constant term

$$\frac{\gamma}{L} + \frac{1}{2} - \frac{1}{L} + \frac{1}{L}\big(p^\tau + \log(1-p^\tau)\big) + C(R; q, \tau) \tag{10}$$

above, using $N = 20$ terms to approximate the value of $C(R; q, \tau)$.

The same recipy allows us to work out the other quantities that we considered at the beginning of the section.

1. The expected number I_n of null rounds, for which we have
$$I_n = 1 - \frac{p}{L} + \frac{1}{L}\big(p^{\tau+1} + \log(1-p^{\tau+1})\big) + C(I; q, \tau) + \delta_I(\log_Q n) + O(1/n).$$

2. The expected total number F_n of coin flips satisfies
$$F_n = \frac{n}{p} + O(1).$$
The main contribution here comes from the pole at $s = -1$. It is worth mentioning that, as we are using the most basic asymptotic estimate $F_n \sim \hat{F}(n)(1 + O(1/n))$ with
$$\hat{F}(z) = \sum_{s_0 \text{ is a pole}} \operatorname{Res}(F^*(s)z^{-s}, s = s_0),$$
the contributions from the poles at $s = 0$ and $s = \chi_j$ are "masked" by the relative error $O(1/n)$ introduced by depoissonization; using more refined versions of analytic depoissonization we could get precise estimations of the lower order terms in F_n.

3. The expected number of leftovers L_n is given by
$$L_n = \frac{1}{L}\left(\frac{1}{1-p^\tau} - \frac{1}{1-p^{\tau+1}} - p^\tau + p^{\tau+1}\right) + C(L; q, \tau) + \delta_L(\log_Q n) + O(1/n).$$

The functions $\delta_I(x)$ and $\delta_L(x)$ above are periodic functions of period 1 and relatively "small" amplitude that collect all the contributions coming from the infinitely many poles $s = \chi_j$ at the imaginary axis.

4 Cumulated null rounds

In this last section, we briefly sketch the approach that we can use to analyze the case where the protocol stops if there have been τ null rounds already, whether consecutive or not.

If we call $S_{n,t}$ the probability of successfully choosing a leader among n players, with failure if there are t null rounds, then

$$S_{n,t} = \sum_{j=1}^{n} \binom{n}{j} p^{n-j} q^j S_{j,t} + p^n S_{n,t-1}, \qquad n \geq 2,$$

with $S_{1,t} = 1$, and $S_{n,0} = 0$ if $n \geq 2$. This recurrence cannot be unwinded like we did to get (2), so we introduce

$$S_t(z) = \sum_{n>0} \frac{S_{n,t}}{n!} z^n.$$

Then, following the same steps as in previous sections

$$\begin{aligned} S_t^*(s) &= \mathcal{M}\left\{e^{-z} S_t(z); s\right\} \\ &= p\frac{\Gamma(s+1)}{1-q^{-s}} + \frac{\mathcal{M}\left\{e^{-z} S_{t-1}(pz); s\right\}}{1-q^{-s}}. \end{aligned}$$

From there we get

$$S_{n,t} \sim \frac{1}{L}\left(p + \sum_{k>0} S_{k,t-1} \frac{p^k}{k}\right) + \delta'(\log_Q n).$$

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