The perimeter of uniform and geometric words: a probabilistic analysis

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Abstract

Let a word be a sequence of \( n \) i.i.d. integer random variables. The perimeter \( P \) of the word is the number of edges of the word, seen as a polyomino. In this paper, we present a probabilistic approach to the computation of the moments of \( P \). This is applied to uniform and geometric random variables. We also show that, asymptotically, the distribution of \( P \) is Gaussian and, seen as a stochastic process, the perimeter converges in distribution to a Brownian motion.

Keywords: Words, perimeter, moments, probabilistic approach, Gaussian distribution, Brownian motion

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1 Introduction

Our attention was recently attracted by a paper by Blecher et al. [4] on the perimeter of words: a word is a sequence of \( n \) i.i.d. integer random variables (RV) \( \{x_0, x_1, \ldots, x_m\}, m := n - 1 \). In [4], the RV are distributed uniformly on \([1, k]\). These RV are also used in this paper. The perimeter \( P_n \) of the word is the number of edges of the word, seen as a polyomino. A typical polyomino, based on the word \( 2, 3, 1, 3, n = 4, P = 18 \) is given in Fig.1.

The mean \( M_{P,n} := \mathbb{E}(P_n) \) and variance \( \mathbb{V}(P_n) \) of \( P_n \) are given in [4], with \( M = \frac{(k-1)(k+1)}{3k} \) by the following theorem:

**Theorem 1.1** In the uniform \([1, k]\) case, \( M_{P,n} \) and \( \mathbb{V}(P_n) \) are given in [4] by

\[
M_{P,n} = (n - 1)M + 2n + (k + 1) = \frac{(3k + 2k^2 + 1) + (k^2 + 6k - 1)n}{3k},
\]

\[
\mathbb{V}(P_n) = \frac{(-5k^2 + 4k^4 + 1) + (-3 + 3k^4)n}{45k^2}.
\]

Some years ago, we had been interested in some uniformly distributed words: see [10]. Moreover, we had analyzed some polyominoes, for instance in [7] and [8], where, in particular, we had derived some limiting Brownian motion (BM) Processes for trajectories. Some recent papers on polyomino’s perimeter are, for instance, [5], [6].

Another classical distribution is the classical geometric(\( p \)) one, with distribution \( pq^{i-1}, i \geq 1, p \in (0, 1), q := 1 - p \). In several papers (some of them with H. Prodinger) we had analyzed related word parameters from a probabilistic point of view. Our last papers on this topic being [13], [12]. We again derived some limiting BM processes, for instance in [9], [11]. For other recent papers on geometric words, see [1], [2].

In the present paper, our motivation is to present a novel approach to the words perimeter problem:

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Figure 1: The polyomino based on the word 2, 3, 1, 3, $n = 4$, $P = 18$

- a probabilistic approach easily leads to the moments of $P_n$,
- the distribution of $P_n$ is asymptotically shown to be Gaussian,
- seen as a stochastic process, the perimeter converges in distribution to a BM,
- our technique is applied to the geometric($p$) case.

2 The mean and variance of $P_n$ in the uniform $[1, k]$ case

In this section, we present a probabilistic approach to the mean and variance of the full perimeter $P_n$.

Set $Q_m := \sum_1^m y_i$, $y_i := |x_i - x_{i-1}|$. Clearly, $P_n = Q_m + x_0 + x_m + 2n$. For further use, we define the vertical perimeter $R_n := Q_m + x_0 + x_m$. We see that the $y_i$ are identically distributed, $y_i$ is correlated with $y_{i+1}$, but independent of $y_k$, $k \geq i + 2$.

The following notations and relations will be used throughout the paper:

- $m := n - 1$,
- $\overline{z} := z - E(z)$, for any RV,
- $M := E(y_i)$,
- $M_{Q,m} := E(Q_m) = mM$,
- $M_{R,n} := E(R_n) = M_{Q,m} + 2E(x_0)$,
- $M_{P,n} := E(P_n) = M_{R,n} + 2n$,
- $T_{\alpha,\beta,\gamma,\delta} := E\left( x_0^{\alpha} \cdot y_1^{\beta} \cdot y_2^{\gamma} \cdot y_3^{\delta} \right)$,
- $T_{0,\beta,\gamma,\delta} := E\left( (y_1 - M)^{\beta} \cdot (y_2 - M)^{\gamma} \cdot (y_3 - M)^{\delta} \right)$,
and the same definitions for $T_{\alpha}, T_{\alpha,\beta}, T_{\alpha,\beta,\gamma}$. When an exponent is null, it means the absence of the relevant variable. For instance, $M \equiv T_{0,1}, T_1 = E(x_0) = \frac{k+1}{2}$ for the uniform case. Explicitly, we have

$$T_{\alpha,\beta,\gamma,\delta} = \left[ \sum_{i=1}^{k} \sum_{j=1}^{k} |j-i|^\beta \sum_{\ell=1}^{k} |\ell-j|^\gamma \sum_{r=1}^{k} |r-\ell|^\delta \right] / k^4.$$ 

Let us first compute the distribution of $y_i$: $f(u) := P(y_i = u), u \in [0, k-1]$. Consider first the case $u > 0$. If $x_1 > x_0, u = x_1 - x_0, x_1 = u + x_0$. But $1 \leq x_1 \leq k$, hence $1 \leq x_0 \leq k - u$. So we first have

$$S_1 := \frac{1}{k} \sum_{i=1}^{k-u} P(x_0 = i) = \frac{k-u}{k^2}.$$ 

Next, if $x_1 < x_0, u = x_0 - x_1, x_1 = x_0 - u$, But $1 \leq x_1 \leq k$, hence $1 + u \leq x_0 \leq k$. So

$$S_2 := \frac{1}{k} \sum_{1+u}^{k} P(x_0 = i) = \frac{k-u}{k^2}.$$ 

Finally,

$$f(u) = S_1 + S_2 = \frac{2(k-u)}{k^2}, u > 0.$$ 

In the case $u = 0$, we simply have $f(0) = \frac{1}{k^2} \sum_{1}^{k} 1 = \frac{1}{k}$. A plot of $f(u), k = 6$, is given in Fig.2.

![Figure 2: $f(u), k = 6$](image)

Now we are ready to compute $M$. This is given either by

$$M := T_{0,1} := \sum_{i=1}^{k} \left( \sum_{j=i}^{k} (j-i) + \sum_{j=1}^{i-1} (i-j) \right) / k^2 = \frac{(k-1)(k+1)}{3k},$$

or $\sum_{0}^{k-1} f(u)u$, which of course leads to the same result.
Indeed, which fits with (1).

Some useful expressions will be used in this section. We collect them here.

\[ T_1 = \mathbb{E}(x_0) = \sum_{i=1}^{k} i/k = \frac{k+1}{2}, \]

\[ T_2 = \mathbb{E}(x_0^2) = \sum_{i=1}^{k} i^2/k = \frac{(k+1)(1 + 2k)}{6}, \]

\[ T_{1,1} = \sum_{i=1}^{k} \left( \sum_{j=1}^{k} (j-i) + \sum_{j=1}^{i-1} (i-j) \right)/k^2 = \frac{(k-1)(k+1)^2}{6k}, \]

\[ T_{0,2} = \sum_{i=1}^{k} \left( \sum_{j=1}^{k} (j-i)^2 \right)/k^2 = \frac{(k-1)(k+1)}{6}, \text{ this is also given by } \sum_{u=1}^{k-1} f(u)u^2, \]

\[ \bar{T}_{0,2} = \sum_{i=1}^{k} \left( \sum_{j=1}^{k} (j-i-M)^2 + \sum_{j=1}^{i-1} (i-j-M)^2 \right)/k^2 = \frac{(k-1)(k+1)(k^2 + 2)}{18k^2}, \]

\[ T_{0,1,1} = \sum_{i=1}^{k} \left[ \sum_{j=i}^{k} (j-i) \left( \sum_{\ell=j}^{k} (\ell-j) + \sum_{\ell=1}^{j-1} (j-\ell) \right) + \sum_{j=1}^{i-1} (i-j) \left( \sum_{\ell=j}^{k} (\ell-j) + \sum_{\ell=1}^{j-1} (j-\ell) \right) \right]/k^3 \]

\[ = \frac{(k-1)(k+1)(7k^2 - 8)}{60k^2}, \]

\[ T_{0,1,1} = \sum_{i=1}^{k} \left[ \sum_{j=i}^{k} (j-i-M) \left( \sum_{\ell=j}^{k} (\ell-j-M) + \sum_{\ell=1}^{j-1} (j-\ell-M) \right) \right]/k^3 \]

\[ = \frac{(k-1)(k-2)(k+2)(k+1)}{180k^2}. \]

These expressions are the only necessary ones in order to compute \( \mathbb{V}(P_n) \).

Now we turn to the computation of variance \( \mathbb{V}(P_n) \). Of course, only \( R_n \) has to be used here. The dominant term of \( \mathbb{V}(R_n) \) is immediate: this is given by

\[ n[(T_{0,2} - M^2) + 2(T_{0,1,1} - M^2)] = nV^*, \]

\[ V^* = \frac{(k-1)(k+1)(k^2 + 1)}{15k^2}. \]

Indeed,

\[ \bar{R}_n = \bar{x}_0 + \bar{x}_m + \bar{Q}_m, \]
and the effect of \( x_0 \) on the variance is just \( \bar{T}_2 + 2\bar{T}_{1,1} = O(1) \). Similarly for the contribution of \( x_m \). Also the contribution of the couples \( y_i y_{i+1} \) is given by \( 2(m-1)\bar{T}_{0,1,1} = 2m\bar{T}_{0,1,1} + O(1) \) and all other contributions are null by independence. \( V^* \) is of course also given by \( \bar{T}_{0,2} + 2\bar{T}_{0,1,1} \).

To compute \( V(P_n) \), we must collect all necessary terms. We symbolically expand

\[
(x_0 + x_m + y_1 + y_i + y_{i+1} + y_{i+2} + y_m)^2.
\]

We collect the relevant contributions, with their weights (we just have to count the corresponding tuples, and, as explained above, all other tuples do not contribute to the variance):

\[
\begin{align*}
y_1^2 & \rightarrow m\bar{T}_{0,2}, \\
y_i y_{i+1} & \rightarrow 2(m-1)\bar{T}_{0,1,1}, \\
x_0^2 & \rightarrow 2\bar{T}_2, \\
x_0 x_m & \rightarrow 2\bar{T}_{1,1}, \\
y_i y_j, i+2 \leq j \leq m, 1 \leq i \leq m-2, & \rightarrow (m-1)(m-2)M^2, \text{ independent RV} \\
x_0 y_1, x_m y_m & \rightarrow 4\bar{T}_{1,1}, \\
x_0 y_i, i > 1, x_m y_j, j < m & \rightarrow 4\bar{T}_1 M(m-1), \text{ independent RV}.
\end{align*}
\]

This gives

\[
V(P_n) = (n-1)\bar{T}_{0,2} + 2\bar{T}_2 + (n-2)2\bar{T}_{0,1,1} + 4\bar{T}_{1,1} + (n-2)(n-3)M^2 + 2\bar{T}_{1}^2 + 4\bar{T}_1 M(n-2) - M_{R,n}^2
\]

\[
= \frac{(-5k^2 + 4k^4 + 1) + (-3 + 3k^4)n}{45k^2},
\]

which fits with (2).

3 \ The third centered moment \( \mu_3(P_n) \) of \( P_n \) in the uniform \([1, k]\) case

In this section, we apply our probabilistic technique to the third centered moment computation. We will only compute the \( n \)--dominant term of \( \mu_3(P_n) \), the complete analysis goes as in Sec. 2, only with elementary but tedious algebra, we omit the details.

The necessary expressions are as follows (for the sake of completeness, we also provide the centered moments):

\[
\bar{T}_{0,3} = \sum_{i=1}^{k} \left( \sum_{j=i}^{k} (j - i - M)^3 + \sum_{j=1}^{i-1} (i - j - M)^3 \right) / k^2 = \frac{(k-1)(k-2)(k+2)(k+1)(2k^2 - 5)}{270k^3},
\]

\[
T_{0,3} = \sum_{u=1}^{k} f(u)u^3 = \sum_{i=1}^{k} \left( \sum_{j=i}^{k} (j - i)^3 + \sum_{j=1}^{i-1} (i - j)^3 \right) / k^2 = \frac{(k-1)(k+1)(3k^2 - 2)}{30k},
\]

\[
\bar{T}_{0,1,1,1} = \sum_{i=1}^{k} \sum_{j=i}^{k} (j - i - M) \left[ \sum_{\ell=j}^{k} (\ell - j - M) \left( \sum_{r=\ell}^{k} (r - \ell - M) + \sum_{r=1}^{\ell-1} (\ell - r - M) \right) \right]
+ \sum_{\ell=1}^{j-1} (j - \ell - M) \left[ \sum_{r=\ell}^{k} (r - \ell - M) + \sum_{r=1}^{\ell-1} (\ell - r - M) \right]
\]

5
\[ + \sum_{j=1}^{i-1} (i - j - M) \left[ \sum_{\ell=j}^{k} (\ell - j - M) \left( \sum_{r=\ell}^{k} (r - \ell - M) + \sum_{r=1}^{\ell-1} (\ell - r - M) \right) \right. \\
\left. + \sum_{\ell=1}^{j-1} (j - \ell - M) \left( \sum_{r=\ell}^{k} (r - \ell - M) + \sum_{r=1}^{\ell-1} (\ell - r - M) \right) \right] / k^4 = -\frac{(k-1)(k-2)(k+2)(k+1)(k^2+5)}{3780k^3}, \]

\[ T_{0,1,1,1} = \sum_{i=1}^{k} \left[ \sum_{j=i}^{k} (j - i - M) \left( \sum_{\ell=j}^{k} (\ell - j - M) \left( \sum_{r=\ell}^{k} (r - \ell - M) + \sum_{r=1}^{\ell-1} (\ell - r - M) \right) \right. \right. \\
\left. \left. + \sum_{\ell=1}^{j-1} (j - \ell - M) \left( \sum_{r=\ell}^{k} (r - \ell - M) + \sum_{r=1}^{\ell-1} (\ell - r - M) \right) \right] \right] / k^4 = \frac{(k-1)(k+1)(17k^4 - 39k^2 + 24)}{420k^3}, \]

\[ T_{0,1,2} = \sum_{i=1}^{k} \left[ \sum_{j=i}^{k} (j - i - M) \left( \sum_{\ell=j}^{k} (\ell - j - M)^2 + \sum_{r=\ell}^{k} (r - \ell - M)^2 \right) \right. \\
\left. + \sum_{\ell=1}^{j-1} (i - j - M) \left( \sum_{r=\ell}^{k} (r - \ell - M)^2 + \sum_{r=1}^{\ell-1} (\ell - r - M)^2 \right) \right] / k^3 = \frac{(k-1)(k-2)(k+2)(k+1)(k^2+2)}{540k^3}, \]

\[ T_{0,1,2} = \sum_{i=1}^{k} \left[ \sum_{j=i}^{k} (j - i) \left( \sum_{\ell=j}^{k} (\ell - j)^2 + \sum_{r=\ell}^{k} (r - \ell)^2 \right) \right. \\
\left. + \sum_{\ell=1}^{j-1} (i - j) \left( \sum_{r=\ell}^{k} (r - \ell)^2 + \sum_{r=1}^{\ell-1} (\ell - r)^2 \right) \right] / k^3 = \frac{(k-1)(k+1)(11k^2 - 14)}{180k}, \]

The couple \( y_i y_{i+1} \) is probabilistically reversible, hence \( T_{0,1,2} = T_{0,2,1} \).

Now we symbolically expand (recall that \( y_i \) is independent of \( y_{i+2} \))

\[ (\overline{y}_i + \overline{y}_{i+1} + \overline{y}_{i+2})^3, \text{ with } \overline{y}_i := y_i - M. \]

Again, the contribution of \( x_0, x_m \) is negligible and terms like \( \overline{y}_i \overline{y}_{i+3} \) lead to 0 by independence. We must only retain the terms

\[ S := \overline{y}_i^3 + 3\overline{y}_i \overline{y}_{i+1} + 3\overline{y}_i^2 \overline{y}_{i+1} + 6\overline{y}_i \overline{y}_{i+1} \overline{y}_{i+2}. \]

Indeed, when counting the tuples, we only retain contributions of order \( m \) and neglect any other \( \mathcal{O}(1) \) terms or null terms (by independence). We expand, this leads to

\[ (y_i^3 + 3y_i y_{i+1}^2 + 6y_i y_{i+1} y_{i+2} + 3y_i^2 y_{i+1}) + (-6y_i^2 - 3y_i^2 - 18y_i y_{i+1} - 6y_i y_{i+1} y_{i+2}) M \]
We make a three steps substitution, *in this order*. For instance, in \( y_i y_{i+1} \), we cannot simply replace \( y_i \) by \( M \) and \( y_{i+1} \) by \( T_{0,2} \). We must use \( T_{0,1,2} \).

- \( y_i^3 = T_{0,3}, y_i^2 y_{i+1} = T_{0,2,1}, y_i y_{i+1}^2 = T_{0,1,2}, y_i y_{i+1} y_{i+2} = T_{0,1,1,1} \)

- \( y_i^2 = T_{0,2}, y_{i+1}^2 = T_{0,2}, y_i y_{i+1} = T_{0,1,1}, y_{i+1} y_{i+2} = T_{0,1,1,2} \)

- \( y_i = M, y_{i+1} = M, y_{i+2} = M \).

This leads to the dominant term of \( \mu_3(P_n) \).

**Theorem 3.1** *In the uniform \([1, k]\) case, the dominant term of \( \mu_3(P_n) \) given by*

\[
\mu_3(P_n) = n \mu_3^* + \mathcal{O}(1),
\]

\[
\mu_3^* = (T_{0,3} + 3T_{0,1,2} + 6T_{0,1,1,1} + 3T_{0,2,1}) + (-9T_{0,2} - 24T_{0,1,1} - 6M^2)M - 66M^3
\]

\[
= \frac{4(k - 2)(1 + 2k)(k + 1)(k - 1)(k + 1)}{945k^3}
\]

Of course, this can also be obtained as

\[
n[T_{0,3} + 3T_{0,1,2} + 6T_{0,1,1,1} + 3T_{0,2,1}],
\]

but we also gave the first approach, which will be used in the next section.

The fourth centered moment \( \mu_4(P_n) \) can be similarly mechanically computed. Note that the dominant term is there of order \( n^2 \): we have contribution of type \( \overline{y_i^2, y_k^2}, k \geq i + 2 \).

### 4 The geometric\((p)\) case

We will now consider the geometric\((p)\) case, with distribution \( pq^{i-1}, i \geq 1, p \in (0, 1), q := 1 - p \). The computation of the centered cross-moments \( \overline{T} \) is rather intricate (in particular with many indices), even for Maple. So we will only use the ordinary cross-moments \( T \). Of course, our techniques can be applied to other polyominoes’ models.

The distribution \( f(u) := \mathbb{P}(y_i = u), u \) is a non-negative integer, is given as follows:

\[
f(u) = \sum_{i=1}^{\infty} \sum_{u=1}^{\infty} pq^{i-1} p^{i+u-1} + \sum_{i=u+1}^{\infty} pq^{i-1} p^{q^{i-u-1}} = \frac{2p(1-p)^u}{2-p}, u > 0,
\]

\[
f(0) = \sum_{i=1}^{\infty} (pq^{i-1} p^{q^{i-1}}) = \frac{p}{2-p}.
\]

A plot of \( f(u), p = 1/2 \) is given in Fig.3.

The first expressions are given as follows

\[
T_1 = \sum_{i=1}^{\infty} pq^{i-1} i = \frac{1}{p},
\]

\[
M = T_{0,1} = \sum_{u=1}^{\infty} f(u) u = \sum_{i=1}^{\infty} pq^{i-1} \left( \sum_{j=i}^{\infty} pq^{j-1}(j - i) + \sum_{j=1}^{i-1} pq^{j-1}(i - j) \right) = \frac{2(1-p)}{p^2(2-p)}.
\]
Figure 3: $f(u), p = 1/2$

hence

$$M_{R,n} = (n - 1)M + 2T_1 = \frac{2 + (2 - 2p)n}{p(2 - p)},$$

$$M_{P,n} = (n - 1)M + 2n + 2T_1 = \frac{2 + (2 + 2p - 2p^2)n}{p(2 - p)}.$$  

We recall a previous definition:

$$T_{\alpha,\beta,\gamma,\delta} := \mathbb{E} \left( x_0^\alpha \cdot y_1^\beta \cdot y_2^\gamma \cdot y_3^\delta \right) = \sum_{i=1}^k \sum_{j=1}^k \sum_{\ell=1}^k |i-j|^\alpha \sum_{r=1}^k |r-\ell|^\beta \cdot |\ell-j|^\gamma \sum_{j=1}^k |r-\ell|^\delta.$$  

The next necessary expressions are given as follows:

$$T_{0,2} = \sum_{i=1}^\infty pq^{i-1} \left( \sum_{j=i}^\infty pq^{i-1}(j-i)^2 + \sum_{j=1}^{i-1} pq^{i-1}(i-j)^2 \right) = \sum_{u=1}^\infty f(u)u^2 = \frac{2(1-p)}{p^2},$$

$$T_{0,3} = \sum_{u=1}^\infty f(u)u^3 = \sum_{i=1}^\infty pq^{i-1} \left( \sum_{j=i}^\infty pq^{i-1}(j-i)^3 + \sum_{j=1}^{i-1} pq^{i-1}(i-j)^3 \right)$$

$$= \frac{2(1-p)(p^2 - 6p + 6)}{p^3(2 - p)},$$

$$T_{0,1,1} = \sum_{i=1}^\infty pq^{i-1} \left[ \sum_{j=i}^\infty pq^{j-1}(j-i) \left( \sum_{\ell=1}^j pq^{\ell-1}(\ell-j) + \sum_{\ell=1}^{j-1} pq^{\ell-1}(j-\ell) \right) \right]$$
\[
T_{0,1,1,1} = \sum_{i=1}^{\infty} pq^{i-1} \left[ \sum_{j=1}^{\infty} pq^{j-1}(j-i) \left( \sum_{\ell=j}^{\infty} pq^{\ell-1}(\ell-j) \left( \sum_{r=\ell}^{\infty} pq^{r-1}(r-\ell) + \sum_{r=1}^{\infty} pq^{r-1}(\ell-r) \right) \right) + \sum_{\ell=1}^{j-1} pq^{\ell-1}(j-\ell) \left( \sum_{r=\ell}^{\infty} pq^{r-1}(r-\ell) + \sum_{r=1}^{\infty} pq^{r-1}(\ell-r) \right) \right] + \sum_{j=1}^{i-1} pq^{j-1}(i-j) \left( \sum_{r=j}^{\infty} pq^{r-1}(r-\ell) \left( \sum_{\ell=1}^{j} pq^{\ell-1}(\ell-j) \right) + \sum_{\ell=1}^{j-1} pq^{\ell-1}(j-\ell) \left( \sum_{r=\ell}^{\infty} pq^{r-1}(r-\ell) \right) \right) \right]
\]

\[
= \frac{2(28 - 84p + 113p^2 - 86p^3 + 39p^4 - 10p^5 + p^6)(1 - p)^2}{p^3(p^2 - 2p + 2)(2 - p)(p^2 + 3 - 3p)^2},
\]

\[
T_{0,1,2} = \sum_{i=1}^{\infty} pq^{i-1} \left[ \sum_{j=1}^{\infty} pq^{j-1}(j-i) \left( \sum_{\ell=j}^{\infty} pq^{\ell-1}(\ell-j)^2 + \sum_{\ell=1}^{j-1} pq^{\ell-1}(j-\ell)^2 \right) \right] + \sum_{j=1}^{i-1} pq^{j-1}(i-j) \left( \sum_{\ell=j}^{\infty} pq^{\ell-1}(\ell-j)^2 + \sum_{\ell=1}^{j-1} pq^{\ell-1}(j-\ell)^2 \right) \right]
\]

\[
= \frac{(28 - 56p + 38p^2 - 10p^3 + p^4)(1 - p)}{p^3(2 - p)^3},
\]

\[
T_{1,1} = \sum_{i=1}^{\infty} pq^{i-1} \left( \sum_{j=1}^{\infty} pq^{j-1}(j-i) + \sum_{j=1}^{i-1} pq^{j-1}(i-j) \right) = \frac{(1 - p)(p^2 - 4p + 6)}{p^2(2 - p)^2}.
\]

Again, \(T_{0,1,2} = T_{0,2,1}\).

The dominant term of \(\mathcal{V}(R_n)\) is given by
(all necessary expressions are extracted from Sec. 2 and 3)
\[
n[(T_{0,2} - M^2) + 2(T_{0,1,1} - M^2)] = nV^*,
\]

\[
V^* = \frac{4(1 - p)(p^4 + 9p^2 - 4p^3 - 10p + 5)}{p^2(2 - p)^2(p^2 + 3 - 3p)}. 
\]

The exact value of \(\mathcal{V}(P_n)\) is given by
\[
\mathcal{V}(R_n) = (n - 1)T_{0,2} + 2T_2 + (n - 2)2T_{0,1,1} + 4T_{1,1} + (n - 2)(n - 3)M^2 + 2T_1^2 + 4T_1M(n - 2) - M_{R,n}^2
\]

\[
= \frac{n[4(1 - p)(p^4 + 9p^2 - 4p^3 - 10p + 5)] + 4(3p^2 - 5p + 5)(1 - p)^2}{p^2(2 - p)^2(p^2 + 3 - 3p)}.
\]
The third centered moment $\mu_3(P_n)$ (dominant term) is given by

$$
\mu_3(P_n) = n[(T_{0,3} + 3T_{0,1,2} + 6T_{0,1,1,1} + 3T_{0,2,1}) - (-9T_{0,2} - 24T_{0,1,1} - 6M^2)M - 26M^3] + O(1) \\
= n\left(8(1-p)(114 - 570p + 1332p^2 - 1908p^3 + 1849p^4 - 1263p^5 + 616p^6 - 213p^7 + 52p^8 - 9p^9 + p^{10})\right) \\
+ O(1).
$$

We summarize our results in the following theorem

**Theorem 4.1** The first three moments of $P_n$ in the geometric($p$) case are given by

$$
M_{P,n} = (n - 1)M + 2n + 2T_1 = \frac{-2 + (-2 - 2p + 2p^2)n}{p(-2 + p)},
$$

$$
\mathbb{V}(P_n) = \frac{n[4(1-p)(p^4 + 9p^2 - 4p^3 - 10p + 5)] + (4p^2 - 5p + 5)(1-p)^2}{p^2(2 - p)^2(2p^2 + 3 - 3p)}
$$

$$
\mu_3(P_n) = n\left(8(1-p)(114 - 570p + 1332p^2 - 1908p^3 + 1849p^4 - 1263p^5 + 616p^6 - 213p^7 + 52p^8 - 9p^9 + p^{10})\right) \\
+ O(1).
$$

## 5 The stochastic processes in the uniform $[1, k]$ case

In this section, we analyze the stochastic processes related to $P_n$. Seen as a stochastic process, the random part of the perimeter is asymptotically given by $Q_m(j) := \sum_{t=1}^{j} y_t$: we can ignore $x_0, x_m$ and the contribution $2n$ is a constant. By the functional central limit theorem ([3, p. 174, Thm. 20.1]), we obtain the following result, where $B(t)$ is the standard Brownian Motion (BM) and $\Rightarrow$ denotes the weak convergence of random functions in the space of all right-continuous functions that have right limits and are endowed with the Skorohod metric (the $\varphi$-mixing property is immediate here: see ([3, p. 167, example 1])). This gives the limiting trajectories corresponding to $Q_m(j)$.

**Theorem 5.1**

$$
\frac{Q_m([mt]) - Mmt}{\sigma \sqrt{m}} \Rightarrow B(t), \quad m \to \infty, \quad t \in [0, 1], \sigma = \sqrt{V^*}.
$$

As a corollary, we have

**Theorem 5.2**

$$
\frac{Q_m - mM}{\sigma \sqrt{m}} \sim \mathcal{N}(0, 1), \quad m \to \infty,
$$

where $\mathcal{N}$ is a Gaussian (normal) random variable.

In the uniform case, $k = 6$, we have made a simulation of $N = 100000$ trajectories $Q_m(j), m = 500$. A typical trajectory is given in Fig.4, together with the drift $jM$. In Fig.5, we show a typical normalized trajectory

$$
\frac{Q_m([mt]) - Mmt}{\sigma \sqrt{m}},
$$

with the classical strongly irregular BM behaviour. We have also computed the observed moments: let $z_t$ denote the $t$th simulated value of $Q_m(m) - mM$. We obtain

$$
\left(\frac{\sum_{t=1}^{N} z_t}{\sigma \sqrt{m}}\right) / N = -0.0038 \ldots , \quad \left(\frac{\sum_{t=1}^{N} \left[\frac{z_t}{\sigma \sqrt{m}}\right]^2}{\sigma \sqrt{m}}\right) / N = 0.991 \ldots \sum_{t=1}^{N} z_t^3 = 1287.47,
$$

to be compared with the theoretical values $\{0, 1, m\mu_3^* = 1569.272976 \ldots \}$. About the third moment, another simulation gives $1911.44 \ldots$: $m$ is not large enough to give a really good fit.
Figure 4: $Q_m(j), m = 500$, drift $= jM$

Figure 5: a typical normalized trajectory
To illustrate Thm 5.2, we have build a histogram as follows: we construct a set of intervals 
\[ I(i) := [i\Delta - 3 - 3\Delta/2, i\Delta - 3 - \Delta/2], \]
i = 0, \ldots, (6/\Delta + 2), centered on \( i\Delta - 3 - \Delta \) and covering the interval \([-3 - \Delta, 3 + \Delta]\). We choose here \( \Delta = 1/2 \). We define cells such that \( cell(i) \) corresponds to interval \( I(i) \). We compute the number \( N[i] \) of values of \( z_\ell \sigma \sqrt{m} \) falling into interval \( I(i) \) and put \( N(i)/N \) into \( cell(i) \) (values < 3.5 are attributed to \( cell(0) \) and similarly for values > 3.5). This gives the empirical histogram. In Fig.6, we compare the cumulative histogram (circle) with the Gaussian distribution function (line): the fit is quite good.

![Figure 6: the cumulative histogram (circle) and the Gaussian distribution function (line)](image)

But it it still more precise to compare, in Fig.7 the histogram itself (circle) with the Gaussian probability mass in interval \( I(i) \): 
\[ \int_{i\Delta - 3 - \Delta/2}^{i\Delta - 3 - 3\Delta/2} \frac{\exp(-x^2/2)}{\sqrt{2\pi}} \, dx \] (line). The fit is quite satisfactory.

We have also made the same kind of simulations for the geometric \( p \) case. The results are quite similar.

6 Conclusion

We have shown that a probabilistic approach leads, almost mechanically, to the first three moments of \( P_n \) and its asymptotic Brownian and Gaussian properties. This technique can be applied to other moments and to other initial probability distributions.

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References

Figure 7: the histogram (circle) and the Gaussian probability mass function in each interval $I(i)$ (line)


