

Two applications of polylog functions, Euler sums and alternating multiple zeta values

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December 25, 2022

Abstract

Let $I(n) := \int_0^1 [x^n + (1-x)^n]^{\frac{1}{n}} dx$. In this paper, we show that $I(n) = \sum_0^\infty \frac{I_i}{n^i}$, $n \rightarrow \infty$ and we compute $I_i, i = 0..10$, obtained by polylog functions, Euler sums and alternating multiple zeta values. We also obtain explicit expressions for some integrals involving functions $u^i, \exp(-u), (1 + \exp(-u))^j, \ln(1 + \exp(-u))^k$. As another asymptotic result, let $S_0(z) := \frac{Li_m(1)}{Li_m(1) - Li_m(z)}$, where $Li_m(z)$ is the polylog function. We provide the asymptotic behaviour of $S_n, n \rightarrow \infty$ where $S_n := [z^n]S_0(z)$. This paper fits within the framework of analytic combinatorics.

Keywords: polylog functions, Euler sums, alternating multiple zeta values, asymptotics, analytic combinatorics

2010 Mathematics Subject Classification: 05A16 60C05 60F05.

1 Introduction

Some time ago, the following question was circulating among the Mathematical problems aficionados: let

$$I(n) := \int_0^1 [x^n + (1-x)^n]^{\frac{1}{n}} dx,$$

what are $I_0 := \lim I(n), I_2 := \lim n^2(I(n) - I_0), n \rightarrow \infty$? I found it interesting to look at a deeper asymptotic analysis of $I(n)$ and found actually that, asymptotically,

$$I(n) = \sum_0^\infty \frac{I_i}{n^i}, n \rightarrow \infty,$$

where I_i are curiously obtained by polylog functions, Euler sums and alternating multiple zeta values. In this paper we compute $I_i, i = 0..10$. We also obtain explicit expressions for some integrals involving functions $u^i, \exp(-u), (1 + \exp(-u))^j, \ln(1 + \exp(-u))^k$ and we propose a conjecture on the general form of I_i . About polylog functions, see de Doelder, [2], Apostol, [1], Lewin, [6], about Euler sums, see Flajolet, Salvy, [3], Xu, [7], and about alternating multiple zeta values, see REFERENCES?

Another problem arose in some work in progress on dynamical systems by Gómez-Aiza and Ward [5]. Ward asked the following question: the polylog function is defined as

$$Li_m(z) := \sum_1^\infty \frac{z^n}{n^m}.$$

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Set

$$S_0(z) := \frac{Li_m(1)}{Li_m(1) - Li_m(z)}$$

and

$$S_n := [z^n]S_0(z).$$

What is the asymptotic behaviour of $S_n, n \rightarrow \infty$? In this paper, we provide the asymptotics of $S_n, m = 3, 4$, up to the $1/n^3$ term. Next terms can be mechanically computed.

2 Analysis of I_n

2.1 A first analysis of I_n with polylog functions

We have

$$I(n) = 2 \int_0^{1/2} [x^n + (1-x)^n]^{\frac{1}{n}} dx = 2 \int_0^{1/2} (1-x) \left[1 + \left(\frac{x}{1-x}\right)^n\right]^{\frac{1}{n}} dx,$$

$$0 \leq \frac{x}{1-x} \leq 1, \text{ let}$$

$$F(n) := \left[1 + \left(\frac{x}{1-x}\right)^n\right]^{\frac{1}{n}} \sim \exp\left[\left(\frac{x}{1-x}\right)^n / n\right],$$

$$\left(\frac{x}{1-x}\right)^n / n \rightarrow 0, n \rightarrow \infty, \text{ exponentially if } x < 1/2, \text{ as } 1/n, \text{ if } x = 1/2.$$

Hence the asymptotic behaviour of $I(n)$ is related to the behaviour of $F(n)$ in the neighbourhood of $x = 1/2$. We set $x = 1/2 - y$ and get

$$I_0 = 2 \int_0^{1/2} (1-x) dx = \frac{3}{4}.$$

We now expand I_n up to the $1/n^5$ term.

$$\begin{aligned} I(n) &= 2 \int_0^{1/2} \left(\frac{1}{2} + y\right) \left[1 + \left(\frac{1-2y}{1+2y}\right)^n\right]^{\frac{1}{n}} dy \\ &= 2 \int_0^{1/2} \left(\frac{1}{2} + y\right) \left[1 + (1-4y+8y^2-16y^3+32y^4 + \mathcal{O}(y^5))^n\right]^{\frac{1}{n}} dy, \text{ and with } y = \frac{u}{4n}, \\ &= \frac{2}{4n} \int_0^{2n} \left(\frac{1}{2} + \frac{u}{4n}\right) \left[1 + (1-u/n + 1/2u^2/n^2 - 1/4u^3/n^3 + 1/8u^4/n^4 + \mathcal{O}(u^5/n^5))^n\right]^{\frac{1}{n}} du \\ &= \frac{2}{4n} \int_0^{2n} \left(\frac{1}{2} + \frac{u}{4n}\right) \left[1 + \exp(-u - 1/12u^3/n^2 + \mathcal{O}(u^5/n^4))\right]^{\frac{1}{n}} du \\ &= \frac{2}{4n} \int_0^{2n} \left(\frac{1}{2} + \frac{u}{4n}\right) \left[1 + \exp(-u) - 1/12 \exp(-u)u^3/n^2 + \exp(-u)\mathcal{O}(u^5/n^4)\right]^{\frac{1}{n}} du \\ &= \frac{2}{4n} \int_0^{2n} \left(\frac{1}{2} + \frac{u}{4n}\right) \exp\left[\ln(1 + \exp(-u))/n - 1/12 \exp(-u)u^3/((1 + \exp(-u))n^3) + \exp(-u)\mathcal{O}(u^5/n^5)\right] du \\ &= \frac{2}{4n} \int_0^{2n} \left(\frac{1}{2} + \frac{u}{4n}\right) \left[1 + \ln(1 + \exp(-u))/n + 1/2 \ln(1 + \exp(-u))^2/n^2\right. \\ &\quad \left.+ 1/12 \left[-\exp(-u)u^3/(1 + \exp(-u)) + 2 \ln(1 + \exp(-u))^3\right]/n^3\right. \\ &\quad \left.+ 1/24 \ln(1 + \exp(-u)) \left[-2 \exp(-u)u^3/(1 + \exp(-u)) + \ln(1 + \exp(-u))^3\right]/n^4 + \exp(-u)\mathcal{O}(u^5/n^5)\right] du \end{aligned}$$

$$\begin{aligned}
&= \frac{3}{4} + \int_0^\infty \left[1/4 \ln(1 + \exp(-u))/n^2 + \left[1/8u \ln(1 + \exp(-u)) + 1/8 \ln(1 + \exp(-u))^2 \right] /n^3 \right. \\
&+ \left. \left[1/16u \ln(1 + \exp(-u))^2 + 1/48 \left[-\exp(-u)u^3/(1 + \exp(-u)) + 2 \ln(1 + \exp(-u))^3 \right] \right] /n^4 \right. \\
&+ 1/96 \left[-\exp(-u)u^4/(1 + \exp(-u)) + 2u \ln(1 + \exp(-u))^3 \right. \\
&- \left. \left. 2 \ln(1 + \exp(-u)) \exp(-u)u^3/(1 + \exp(-u)) + \ln(1 + \exp(-u))^4 \right] /n^5 \right. \\
&\left. + \exp(-u)\mathcal{O}(u^6/n^6) \right] du.
\end{aligned}$$

We immediately recover I_0 . The computation of $I_i, i \geq 1$ is detailed in the next sections.

2.1.1 Computation of I_1, I_2, I_3, I_4

We have

$$I_1 = 0,$$

$$I_2 = \int_0^\infty 1/4 \ln(1 + \exp(-u)) du = 1/4 \sum_1^\infty \frac{(-1)^{i+1}}{i^2} = \frac{\pi^2}{48},$$

$$I_3 = \int_0^\infty [1/8u \ln(1 + \exp(-u)) + 1/8 \ln(1 + \exp(-u))^2] du = 11/32\zeta(3) + 1/8I \ln(2)^2\pi + 1/4 \ln(2)Li_2(2) - 1/4Li_3(2),$$

where the polylog function is defined by

$$Li_n(z) = \sum_1^\infty \frac{z^k}{k^n}.$$

But we know that

$$Li_n(z) = -(-1)^n Li_n(1/z) - \frac{(2\pi I)^n}{n!} B_n \left(\frac{1}{2} + \frac{\ln(-z)}{2\pi I} \right), z \notin [0, 1],$$

where $B_n(x)$ is the n th Bernoulli polynomial, and

$$Li_2(1/2) = 1/12\pi^2 - 1/2 \ln(2)^2, \text{ hence } Li_2(2) = 1/4\pi^2 - I\pi \ln(2),$$

$$Li_3(1/2) = 7/8\zeta(3) - 1/12\pi^2 \ln(2) + 1/6 \ln(2)^3, \text{ hence } Li_3(2) = 7/8\zeta(3) + 1/4\pi^2 \ln(2) - 1/2I\pi \ln(2)^2.$$

The values of $Li_k(1/2), k \geq 4$ are not known to be related to classical constants.

This leads to

$$I_3 = \frac{\zeta(3)}{8}.$$

Another, more elegant, way to compute I_3 is to turn to Euler sums. Following Flajolet, Salvy, [3], we have

$$S_{p,q}^{+-} := \sum_{k=1}^\infty (-1)^{k-1} \frac{H_k^{(p)}}{k^q}, H_n^{(p)} := \sum_{j=1}^n \frac{1}{j^p},$$

$$\bar{\zeta}(s) := (1 - 2^{1-s})\zeta(s), \bar{\zeta}(1) := \ln(2),$$

$$2S_{1,q}^{+-} = (q+1)\bar{\zeta}(q+1) - \zeta(q+1) - 2 \sum_{k=1}^{q/2-1} \bar{\zeta}(k)\zeta(q+1-2k), 1+q \text{ odd},$$

$$2S_{1,2}^{+-} = \frac{5}{4}\zeta(3),$$

$$2S_{1,4}^{+-} = 59/16\zeta(5) - 1/6\pi^2\zeta(3).$$

Hence

$$\int_0^\infty u \ln(1 + \exp(-u)) du = \sum_1^\infty \frac{(-1)^{i+1}}{i^3} = \frac{3}{4}\zeta(3),$$

$$\int_0^\infty \ln(1 + \exp(-u))^2 du = \sum_1^\infty \sum_1^\infty \frac{(-1)^{i+j}}{ij(i+j)} = \sum_{k=2}^\infty \sum_{i=1}^{k-1} \frac{(-1)^k}{ki(k-i)} = \sum_{k=2}^\infty \frac{(-1)^k}{k^2} 2H_{k-1}$$

$$= 2 \sum_{k=1}^\infty \frac{(-1)^k}{k^2} H_k - 2 \sum_{k=1}^\infty \frac{(-1)^k}{k^3} = -2S_{1,2}^{+-} + \frac{3}{2}\zeta(3) = \frac{\zeta(3)}{4}.$$

We immediately recover I_3 .

Another similar sum will be used in the next section: for $p+q$ odd,

$$S_{p,q}^{+-} := \left[(1 - (-1)^p)\zeta(p)\bar{\zeta}(q) + \bar{\zeta}(p+q) + 2 \sum_{k=0}^{\lfloor p/2 \rfloor} \binom{q+p-2k-1}{q-1} (-1)^{p-2k+1} \bar{\zeta}(q+p-2k) \bar{\zeta}(2k) \right. \\ \left. + 2(-1)^p \sum_0^{\lfloor q/2 \rfloor} \binom{p+q-2k-1}{p-1} \zeta(p+q-2k) \bar{\zeta}(2k) \right] / 2,$$

$$S_{2,3}^{+-} = -11/32\zeta(5) + 5/48\zeta(3)\pi^2.$$

The computation of I_4 goes as follows

Now we have

$$S_1 := \int_0^\infty \frac{u^3 e^{-u}}{1 + e^{-u}} du = \frac{7\pi^4}{120},$$

$$S_2 := \int_0^\infty \ln(1 + \exp(-u))^3 du = \ln(2)^3 \pi I + 3 \ln(2)^2 Li_2(2) - 6 \ln(2) Li_3(2) + 6 Li_4(2) - 1/15\pi^4$$

$$= 1/4\pi^2 \ln(2)^2 - 21/4 \ln(2) \zeta(3) - 6 Li_4(1/2) + 1/15\pi^4 - 1/4 \ln(2)^4,$$

$$S_3 := \int_0^\infty u \ln(1 + \exp(-u))^2 du = \sum_{k=2}^\infty \frac{(-1)^k}{k^3} 2H_{k-1} = 2 \sum_{k=1}^\infty \frac{(-1)^k}{k^3} H_k - 2 \sum_{k=1}^\infty \frac{(-1)^k}{k^4} = -2S_{1,3}^{+-} - 2 \left(-\frac{7}{720} \pi^4 \right),$$

$$S_{1,3}^{+-} = -2 Li_4(1/2) + 11/4 \zeta(4) + 1/2 \zeta(2) \ln(2)^2 - 1/12 \ln(2)^4 - 7/4 \zeta(3) \ln(2), \text{ this is } \mu_1 \text{ in [3],}$$

$$S_3 = 4 Li_4(1/2) - 1/24 \pi^4 - 1/6 \pi^2 \ln(2)^2 + 1/6 \ln(2)^4 + 7/2 \ln(2) \zeta(3).$$

Hence

$$I_4 := \frac{1}{48} [-S_1 + 2S_2 + 3S_3] = -\frac{\pi^4}{960}.$$

2.1.2 Computation of I_5

We compute

$$S_4 := \int_0^\infty \frac{u^4 e^{-u}}{1 + e^{-u}} du = 45/2 \zeta(5).$$

Now we turn to S_5 :

$$\begin{aligned}
S_5 &:= \int_0^\infty u \ln(1 + \exp(-u))^3 du = \sum_{i=1}^\infty \sum_{j=1}^\infty \sum_{\ell=1}^\infty \frac{(-1)^{i+j+\ell+1}}{ij\ell(i+j+\ell)^2} = \sum_{k=3}^\infty \frac{(-1)^{k+1}}{k^2} \sum_{v=2}^{k-1} \frac{1}{k-v} \sum_{i=1}^{v-1} \frac{1}{i(v-i)} \\
&= \sum_{v=2}^\infty 2 \frac{H_{v-1}}{v} \sum_{k=v+1}^\infty \frac{(-1)^{k+1}}{k^2(k-v)} \\
&= 2 \sum_{v=2}^\infty \frac{(-1)^{k-1}}{k^2} \sum_{j=1}^{k-1} \frac{H_{j-1}}{j(k-j)} \\
&= 2 \sum_{v=2}^\infty \frac{(-1)^{k-1}}{k^3} \sum_{j=1}^{k-1} H_{j-1} \left[\frac{1}{j} + \frac{1}{k-j} \right] \\
&= \sum_{v=2}^\infty \frac{(-1)^{k-1}}{k^3} \left[H_{k-1}^2 - H_{k-1}^{(2)} + 2H_{k-1}^2 - 2H_{k-1}^{(2)} \right] \\
&= 3 \sum_{v=2}^\infty \frac{(-1)^{k-1}}{k^3} \left[H_{k-1}^2 - H_{k-1}^{(2)} \right].
\end{aligned}$$

But

$$\begin{aligned}
H_{n-1}^2/n^3 &= [H_n^2 - 2H_n/n + 1/n^2]/n^3, \\
H_{n-1}^{(2)}/n^3 &= [H_n^{(2)} - 1/n^2]/n^3.
\end{aligned}$$

Hence

$$S_5 = -3(T_1 + T_2), \text{ with}$$

$$T_2 = - \left[-S_{2,3}^{+-} + 15/16\zeta(5) \right] = 5/48\zeta(3)\pi^2 - \frac{41}{32}\zeta(5),$$

$$T_1 = T_3 + 2S_{1,4}^{+-} - 15/16\zeta(5),$$

$$T_3 = \sum_{k=1}^\infty (-1)^k \frac{H_k^2}{k^3}$$

$$= -(4Li_5(1/2) + 4\ln(2)Li_4(1/2) + 2/15\ln(2)^5 + 7/4\zeta(3)\ln(2)^2$$

$$- 19/32\zeta(5) - 2/3\zeta(2)\ln(2)^3 - 11/8\zeta(2)\zeta(3)), \text{ see [7], where many recent references can be found, so}$$

$$T_1 = -4Li_5(1/2) - 4\ln(2)Li_4(1/2) - 2/15\ln(2)^5 - 7/4\zeta(3)\ln(2)^2 + \frac{107}{32}\zeta(5) + 1/9\pi^2\ln(2)^3 + 1/16\zeta(3)\pi^2,$$

and finally

$$S_5 = 12Li_5(1/2) + 12\ln(2)Li_4(1/2) + 2/5\ln(2)^5 + 21/4\zeta(3)\ln(2)^2 - \frac{99}{16}\zeta(5) - 1/3\pi^2\ln(2)^3 - 1/2\zeta(3)\pi^2,$$

$$\begin{aligned}
S_6 &:= \int_0^\infty \frac{u^3 e^{-u} \ln(1 + \exp(-u))}{1 + e^{-u}} du = \int_0^\infty u^3 \sum_{k=2}^\infty e^{-uk} (-1)^k \sum_{i=1}^{k-1} \frac{1}{i} du = 3! \sum_{k=2}^\infty \frac{1}{k^4} (-1)^k H_{k-1} \\
&= 3! \left[\sum_{k=1}^\infty \frac{1}{k^4} (-1)^k H_k - \sum_{k=1}^\infty \frac{1}{k^5} (-1)^k \right] = 3! \left[-S_{1,4}^{+-} + \frac{15}{16}\zeta(5) \right] = -87/16\zeta(5) + 1/2\pi^2\zeta(3),
\end{aligned}$$

$$S_7 := \int_0^\infty \ln(1 + \exp(-u))^4 du$$

$$= 2/3\pi^2\ln(2)^3 - 21/2\ln(2)^2\zeta(3) - 24\ln(2)Li_4(1/2) - 4/5\ln(2)^5 - 24Li_5(1/2) + 24\zeta(5).$$

Hence

$$I_5 = \frac{1}{96} [-S_4 + 2S_5 - 2S_6 + S_7] = -1/48\zeta(3)\pi^2.$$

Let us summarize our results in the following theorem:

Theorem 2.1 *Let*

$$I(n) := \int_0^1 [x^n + (1-x)^n]^{\frac{1}{n}} dx.$$

We have

$$I(n) = \sum_0^{\infty} \frac{I_i}{n^i}, n \rightarrow \infty, I_0 = \frac{3}{4}, I_1 = 0, I_2 = \frac{\pi^2}{48}, I_3 = \frac{\zeta(3)}{8}, I_4 = -\frac{\pi^4}{960}, I_5 = -1/48\zeta(3)\pi^2.$$

We also have the following corollary, using polylog functions (many other similar results can be found in Xu [7])

Corollary 2.2

$$\begin{aligned} \int_0^{\infty} 1/4 \ln(1 + \exp(-u)) du &= \frac{\pi^2}{48}, \\ \int_0^{\infty} u \ln(1 + \exp(-u)) du &= \frac{3}{4}\zeta(3), \\ \int_0^{\infty} \ln(1 + \exp(-u))^2 du &= \frac{\zeta(3)}{4}, \\ \int_0^{\infty} \frac{u^3 e^{-u}}{1 + e^{-u}} du &= \frac{7\pi^4}{120}, \\ \int_0^{\infty} \ln(1 + \exp(-u))^3 du &= 1/4\pi^2 \ln(2)^2 - 21/4 \ln(2)\zeta(3) - 6Li_4(1/2) + 1/15\pi^4 - 1/4 \ln(2)^4, \\ \int_0^{\infty} u \ln(1 + \exp(-u))^2 du &= 4Li_4(1/2) - 1/24\pi^4 - 1/6\pi^2 \ln(2)^2 + 1/6 \ln(2)^4 + 7/2 \ln(2)\zeta(3), \\ \int_0^{\infty} \frac{u^4 e^{-u}}{1 + e^{-u}} du &= 45/2\zeta(5), \\ \int_0^{\infty} u \ln(1 + \exp(-u))^3 du \\ &= 12Li_5(1/2) + 12 \ln(2) Li_4(1/2) + 2/5 \ln(2)^5 + 21/4 \zeta(3) \ln(2)^2 - \frac{99}{16} \zeta(5) - 1/3 \pi^2 \ln(2)^3 - 1/2 \zeta(3) \pi^2, \\ \int_0^{\infty} \frac{u^3 e^{-u} \ln(1 + \exp(-u))}{1 + e^{-u}} du &= -87/16 \zeta(5) + 1/2 \pi^2 \zeta(3), \\ \int_0^{\infty} \ln(1 + \exp(-u))^4 du \\ &= 2/3 \pi^2 \ln(2)^3 - 21/2 \ln(2)^2 \zeta(3) - 24 \ln(2) Li_4(1/2) - 4/5 \ln(2)^5 - 24 Li_5(1/2) + 24 \zeta(5). \end{aligned}$$

We now have two problem:

problem 1: how to explain the relatively simple I_i expressions, depending only on ordinary zeta values ?

problem 2: can we find ‘easily’ similar computations for $I_i, i \geq 6$?

These problems will be considered in the next sections.

Let

$$\begin{aligned} I_{n,2} &:= I_0 + I_2/n^2, \\ I_{n,3} &:= I_0 + I_2/n^2 + I_3/n^3, \\ I_{n,4} &:= I_0 + I_2/n^2 + I_3/n^3 + I_4/n^4. \end{aligned}$$

To check the quality of our asymptotics, we display, in Figure 1, $I_0, I(n), I_{n,2}, I_{n,3}, I_{n,4}$.

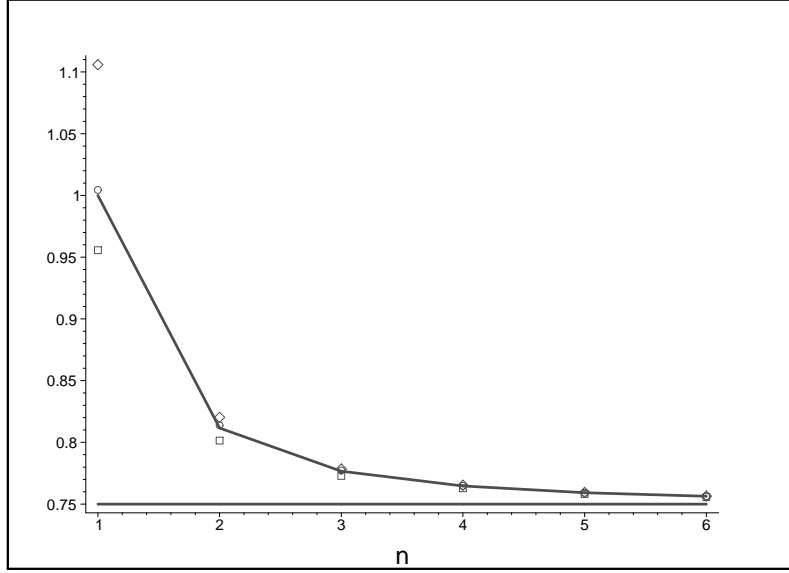


Figure 1: $I(n)$ (line), I_0 (line), $I_{n,2}$ (box), $I_{n,3}$ (diamond), $I_{n,4}$ (circle)

2.2 Analysis of I_n : Hofmann's approach

2.3 Analysis of I_n : Kuba's approach

3 Analysis of S_n

3.1 A first approach

We know that

$$Li_m(z) = \frac{(-1)^m}{(m-1)!} w^{m-1} (\ln(w) - H_{m-1}) + \sum_{j \neq m-1, j \geq 0} \frac{(-1)^j}{j!} \zeta(m-j) w^j, \text{ see [4], VI.20, with } \quad (1)$$

$$w := -\ln(z).$$

The singularity $z = 1$ in $S_0(z)$ leads to the desired expansion of S_n . Set

$$L := \ln \left(\frac{1}{1-z} \right) = Li_1(z),$$

$$L_{k,n} := [z^n] L^k, \text{ with}$$

$$L_{1,n} = \frac{1}{n}.$$

Let

$$\varepsilon := 1 - z,$$

$$S(\varepsilon, L) := \text{expansion of } S_0(1 - \varepsilon) \text{ w.r.t. } \varepsilon,$$

we compute successively

$$D_{i,j} := [\varepsilon^i L^j] S(\varepsilon, L), \text{ which depends on } m,$$

$$\begin{aligned}
G_{i,j,n} &:= [z^n] \varepsilon^i L^j, \text{ which is independent of } m, \\
T_{i,j,n} &= D_{i,j} G_{i,j,n} = [z^n \varepsilon^i L^j] S(\varepsilon, L), \\
S_n &:= [z^n] S_0(z), \\
T_n &:= \text{asymptotics of } S_n = \sum_i \sum_j T_{i,j,n}, n \rightarrow \infty.
\end{aligned}$$

We also define

$$C_{n,k} := \text{asymptotics of } S_n \text{ up to the } 1/n^k \text{ term.}$$

In this paper, we will compute $C_{n,k}, k = 0, 3, m = 3, 4$, but as we will see, more terms can be mechanically obtained. We will also show some graphs of $C_{n,k}$.

3.2 Some asymptotics for $L_{k,n}$

Set

$$H := \Gamma(n + \alpha) / (\Gamma(\alpha) \Gamma(n + 1)), \text{ see [1], VI.7,}$$

$$\frac{\partial^2 H(\alpha)}{\partial \alpha^2} = [\psi(1, n + \alpha) + \psi(n + \alpha)^2 - 2\psi(n + \alpha)\psi(\alpha) + \psi(\alpha)^2 - \psi(1, \alpha)] \Gamma(n + \alpha) / (\Gamma(\alpha) \Gamma(n + 1)),$$

where $\psi(n, x)$ is the n th polygamma function, which is the n th derivative of the digamma function, we have

$$L_{2,n} = \lim_{\alpha \rightarrow 0} \frac{\partial^2 H(\alpha)}{\partial \alpha^2} = (2\psi(n) + 2\gamma) / n,$$

$$L_{2,n} = (2 \ln(n) + 2\gamma) / n - 1/n^2 - 1/(6n^3) + 1/(60n^5) + \mathcal{O}(1/n^6), \text{ see [1], Figure VI.5 for the first terms,}$$

$$\begin{aligned}
\frac{\partial^3 H(\alpha)}{\partial \alpha^3} &= [\psi(2, n + \alpha) + 3\psi(1, n + \alpha)\psi(n + \alpha) - 3\psi(1, n + \alpha)\psi(\alpha) \\
&\quad + \psi(n + \alpha)^3 - 3\psi(n + \alpha)^2\psi(\alpha) + 3\psi(n + \alpha)\psi(\alpha)^2 \\
&\quad - 3\psi(n + \alpha)\psi(1, \alpha) - \psi(\alpha)^3 + 3\psi(\alpha)\psi(1, \alpha) - \psi(2, \alpha)] \Gamma(n + \alpha) / (\Gamma(\alpha) \Gamma(n + 1)),
\end{aligned}$$

$$L_{3,n} = \lim_{\alpha \rightarrow 0} \frac{\partial^3 H(\alpha)}{\partial \alpha^3} = 1/2(6\psi(1, n) + 12\psi(n)\gamma - \pi^2 + 6\gamma^2 + 6\psi(n)^2) / n,$$

$$\begin{aligned}
L_{3,n} &= (3 \ln(n)^2 + 6 \ln(n)\gamma - 1/2\pi^2 + 3\gamma^2) / n + (3 - 3 \ln(n) - 3\gamma) / n^2 + (-1/2\gamma + 9/4 - 1/2 \ln(n)) / n^3 \\
&\quad + 3/(4n^4) + (1/20\gamma + 1/48 + 1/20 \ln(n)) / n^5 + \mathcal{O}(1/n^6).
\end{aligned}$$

3.3 The case $m = 3, 4$

We have, for $m = 3$, by (1),

$$Li_3(z) = -1/2w^2(\ln(w) - 3/2) + \zeta(3) - 1/6\pi^2w + 1/12w^3 - 1/288w^4 + 1/86400w^6 - 1/10160640w^8 + \mathcal{O}(w^9),$$

$$S_0(z) = \zeta(3) / (1/2w^2(\ln(w) - 3/2) + 1/6\pi^2w - 1/12w^3 + 1/288w^4 - 1/86400w^6 + 1/10160640w^8 + \mathcal{O}(w^9)),$$

we have the expansions

$$w = \varepsilon + 1/2\varepsilon^2 + 1/3\varepsilon^3 + 1/4\varepsilon^4 + \mathcal{O}(\varepsilon^5),$$

$$\ln(w) = -L + 1/2\varepsilon + 5/24\varepsilon^2 + 1/8\varepsilon^3 + \mathcal{O}(\varepsilon^4).$$

Hence

$$\begin{aligned}
S(\varepsilon, L) &= 6\zeta(3) / (\pi^2\varepsilon) + 1/8\zeta(3)(-24\pi^6 + 144\pi^4L + 216\pi^4) / \pi^8 \\
&\quad + 1/8\zeta(3)(-4\pi^6 - 48\pi^4 + 432\pi^2L^2 + 1296\pi^2L + 972\pi^2) / \pi^8\varepsilon \\
&\quad + 1/8\zeta(3)(-2\pi^6 - 19\pi^4 + 360\pi^2L + 216\pi^2L^2 + 54\pi^2 + 1296L^3 + 5832L^2 + 8748L + 4374) / \pi^8\varepsilon^2 + \mathcal{O}(\varepsilon^3).
\end{aligned}$$

This leads successively to

$$\begin{aligned}
D_{-1,0} &= 6\zeta(3)/\pi^2, \\
D_{0,1} &= 18\zeta(3)/\pi^4, \\
D_{1,1} &= 162\zeta(3)/\pi^6, \\
D_{1,2} &= 54\zeta(3)/\pi^6,
\end{aligned}$$

$$T_{-1,0,n} = 6\zeta(3)/\pi^2, T_{-1,k,n} = 0, k > 0,$$

$$T_{0,1,n} = 18\zeta(3)/(\pi^4 n), T_{0,k,n} = 0, k > 1,$$

The general form of $G_{i,j}$ is given in [4], Equ. (27). The detailed computation goes as follows,

$$G_{1,1,n} = L_{1,n} - L_{1,n-1} = -1/n^2 - 1/n^3 - 1/n^4 - 1/n^5 + \mathcal{O}(1/n^6),$$

$$T_{1,1,n} = D_{1,1}G_{1,1,n} = -162\zeta(3)/(\pi^6 n^2) - 162\zeta(3)/(\pi^6 n^3) + \mathcal{O}(1/n^4),$$

$$\begin{aligned}
G_{1,2,n} &= L_{2,n} - L_{2,n-1} = (2 - 2\ln(n) - 2\gamma)/n^2 + (5 - 2\ln(n) - 2\gamma)/n^3 \\
&\quad + (43/6 - 2\ln(n) - 2\gamma)/n^4 + (55/6 - 2\ln(n) - 2\gamma)/n^5 + \mathcal{O}(1/n^6),
\end{aligned}$$

$$T_{1,2,n} = D_{1,2}G_{1,2,n} = 54(2 - 2\ln(n) - 2\gamma)\zeta(3)/(\pi^6 n^2) + 54(5 - 2\ln(n) - 2\gamma)\zeta(3)/(\pi^6 n^3) + \mathcal{O}(1/n^4),$$

$$D_{2,3} = 162\zeta(3)/\pi^8,$$

$$D_{2,2} = 27\zeta(3)(27 + \pi^2)/\pi^8,$$

$$D_{2,1} = 9/2\zeta(3)(10\pi^2 + 243)/\pi^8,$$

$$G_{2,3,n} = L_{3,n} - 2L_{3,n-1} + L_{3,n-2} = (6 - 18\gamma - \pi^2 - 18\ln(n) + 6\gamma^2 + 6\ln(n)^2 + 12\ln(n)\gamma)/n^3 + \mathcal{O}(1/n^4),$$

$$G_{2,2,n} = L_{2,n} - 2L_{2,n-1} + L_{2,n-2} = (-6 + 4\ln(n) + 4\gamma)/n^3 + \mathcal{O}(1/n^4),$$

$$G_{2,1,n} = L_{1,n} - 2L_{1,n-1} + L_{1,n-2} = 2/n^3 + \mathcal{O}(1/n^4),$$

$$T_{2,3,n} = D_{2,3}G_{2,3,n} = 162(6 - 18\gamma - \pi^2 - 18\ln(n) + 6\gamma^2 + 6\ln(n)^2 + 12\ln(n)\gamma)\zeta(3)/(n^3\pi^8) + \mathcal{O}(1/n^4),$$

$$T_{2,2,n} = D_{2,2}G_{2,2,n} = 27(-6 + 4\ln(n) + 4\gamma)\zeta(3)(27 + \pi^2)/(n^3\pi^8) + \mathcal{O}(1/n^4),$$

$$T_{2,1,n} = D_{2,1}G_{2,1,n} = 9\zeta(3)(10\pi^2 + 243)/(n^3\pi^8) + \mathcal{O}(1/n^4).$$

Finally

$$T_n = T_{-1,0,n} + T_{0,1,n} + \sum_i \sum_j T_{i,j,n}.$$

This leads to the following theorem:

Theorem 3.1

Let

$$S_0(z) := \frac{Li_3(1)}{Li_3(1) - Li_3(z)}, \text{ then}$$

$$\begin{aligned}
S_n &:= [z^n]S_0(z) = 6\zeta(3)/\pi^2 + 18\zeta(3)/(\pi^4 n) + 3\zeta(3)(-18\pi^2 - 36\pi^2 \ln(n) - 36\pi^2 \gamma)/(\pi^8 n^2) \\
&\quad + 3\zeta(3)(-42\pi^2 - 405 + 324\gamma^2 + 324\ln(n)^2 + 648\ln(n)\gamma)/(\pi^8 n^3) + \mathcal{O}(1/n^4).
\end{aligned}$$

This gives

$$C_{n,0} = 6\zeta(3)/\pi^2,$$

$$C_{n,1} = 6\zeta(3)/\pi^2 + 18\zeta(3)/(\pi^4 n),$$

$$C_{n,2} = 6\zeta(3)/\pi^2 + 18\zeta(3)/(\pi^4 n) + 3\zeta(3)(-18\pi^2 - 36\pi^2 \ln(n) - 36\pi^2 \gamma)/(\pi^8 n^2),$$

$$\begin{aligned}
C_{n,3} &= 6\zeta(3)/\pi^2 + 18\zeta(3)/(\pi^4 n) + 3\zeta(3)(-18\pi^2 - 36\pi^2 \ln(n) - 36\pi^2 \gamma)/(\pi^8 n^2) \\
&\quad + 3\zeta(3)(-42\pi^2 - 405 + 324\gamma^2 + 324\ln(n)^2 + 648\ln(n)\gamma)/(\pi^8 n^3).
\end{aligned}$$

To check the quality of our asymptotics, we display, in Figure 2, $S_n, C_{n,0}, C_{n,1}, C_{n,2}, C_{n,3}$.

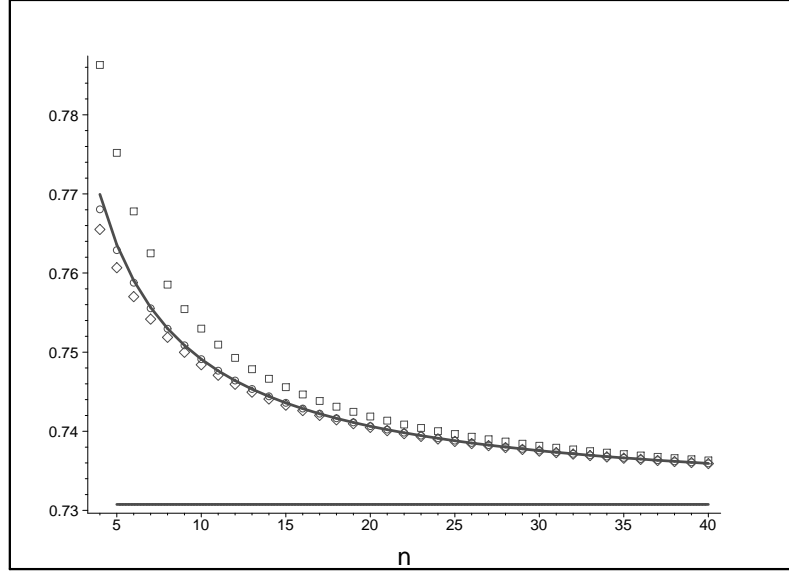


Figure 2: $m = 3$, S_n (line), $C_{n,0}$ (line), $C_{n,1}$ (box), $C_{n,2}$ (diamond), $C_{n,3}$ (circle)

The convergence of S_n to $C_{n,0}$ is rather slow: we have $C_{n,0} = 0.7307629692\dots$ and $S_{100} = 0.7329\dots$

The case $m = 4$ is mechanically treated like the case $m = 3$. We obtain

$$\begin{aligned} C_{n,0} &= 1/90\pi^4/\zeta(3), \\ T_{0,1,n} &= 0, \\ C_{n,2} &= 1/90\pi^4/\zeta(3) + 1/540\pi^4/(\zeta(3)^2n^2), \\ C_{n,3} &= 1/90\pi^4/\zeta(3) + 1/540\pi^4(\zeta(3)^2n^2) - 1/1620\pi^6/(\zeta(3)^3n^3), \end{aligned}$$

we display, in Figure 3, $S_n, C_{n,0}, C_{n,2}, C_{n,3}$. The convergence of S_n to $C_{n,0}$ is faster.

Remarks

1. in order to expand $S_0(1 - \varepsilon)$, we first expand w.r.t ε as $\varepsilon = o(1/L^k), k > 0$
2. $G_{k,j}$ starts with a $1/n^{k+1}$ term, which allows an easy expansion
3. more and more terms in the expansion of $S(\varepsilon, L)$ are needed when m increases: the first terms don't contain any L terms. For instance, for $m = 6$, only the ε^3 contains a linear L contribution, and the asymptotics of S_n starts as $1/945\pi^6/\zeta(5) + 1/18900\pi^6/(\zeta(5)^2n^4)$. More terms can be mechanically computed.

4 Acknowledgements

We would like to thank H.Prodinger for helping in computing an Euler sum and W.Wang for providing a useful reference.

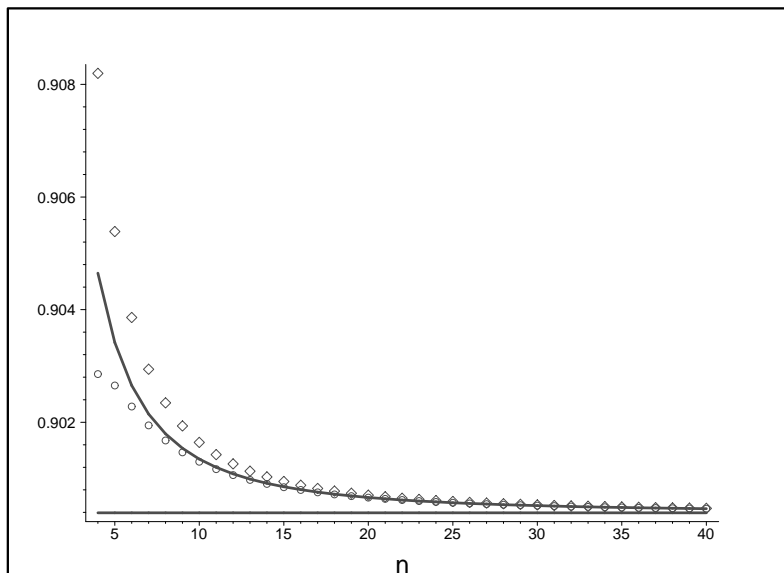


Figure 3: $m = 4$, S_n (line), $C_{n,0}$ (line), $C_{n,2}$ (diamond), $C_{n,3}$ (circle)

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