Representations of numbers as $\sum_{k=-n}^{n} \varepsilon_k k$: A saddle point approach

Guy Louchard* and Helmut Prodinger†

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Abstract

Using the saddle point method, we obtain from the generating function of the numbers in the title and Cauchy's integral formula asymptotic results of high precision in central and non-central regions.

Keywords: Representations of integers, saddle point method, asymptotic expansion

1 Introduction

We consider the number of representations of m as $\sum_{k=-n}^{n} \varepsilon_k k$, where $\varepsilon_k \in \{0, 1\}$. For $m = 0$, this is sequence A000980 in Sloane's encyclopedia. This problem has a long history: Van Lint [7] found an asymptotic formula for the number of representations of 0, Entringer [2] did the same for a fixed number m. Clark [1] allowed m to be $o(m^{3/2})$.

Here, we extend the range a bit, to $\mathcal{O}(m^{3/2})$. But we improve at the same time the quality of the approximation. We are doing this using the saddle point method. Implicitly, in the earlier writings, it was also used, but $z = 1$ was taken as an approximative saddle point, whereas we employ better approximations, to get more precise results.

We are also able to choose m in a non-central region. As a showcase, we consider $m = n^{7/4}$, but we could deal with other values as well.

Our main findings are the asymptotic formulæ (6) (central range) and (7) (non-central range).

The generating function of the number of representations for fixed n is given by

$$C_n(z) = 2 \prod_{k=1}^{n} (1 + z^k)(1 + z^{-k})$$

and

$$C_n(1) = 2 \cdot 4^n.$$

*Université Libre de Bruxelles, Département d'Informatique, CP 212, Boulevard du Triomphe, B-1050 Bruxelles, Belgium, email: louchard@ulb.ac.be

†University of Stellenbosch, Mathematics Department, 7602 Stellenbosch, South Africa, email: hproding@sun.ac.za

By normalisation, we get the probability generating function of a random variable X_n:

$$F_n(z) = 4^{-n} \prod_{k=1}^{n} (1+z^k)(1+z^{-k}),$$

from which we obtain mean $\mathbb{M}$ and variance $\mathbb{V}$:

$$\mathbb{M}(n) = 0, \quad \sigma^2 := \mathbb{V}(n) = \frac{n(n+1)(2n+1)}{12}.$$

By Cauchy's theorem,

$$P_n(j) = \mathbb{P}(X_n = j) = \frac{1}{2\pi i} \int_{\mathcal{C}} F_n(z) \frac{dz}{z^{j+1}},$$

where $\mathcal{C}$ is, say, a circle around the origin.

2 The Gaussian limit – approximation near the mean

We consider values $j = x\sigma$, for $x = \mathcal{O}(1)$ in a neighbourhood of the mean 0.

The Gaussian limit of X_n can be obtained by using the Lindeberg-Lévy conditions, see for instance, Feller [3], but we want more precision.

We know that

$$P_n(j) = \frac{1}{2\pi i} \int_{\Omega} e^{S(z)} dz$$

where

$$S(z) := \ln(F_n(z)) - (j+1) \ln z$$

with

$$\ln(F_n(z)) = \sum_{i=1}^{n} [\ln(1+z^i) + \ln(1+z^{-i}) - \ln 4].$$

Since $F_n(z)$ is a (Laurent-)polynomial, the analyticity restriction can be ignored. We will use as Ω a circle around the origin passing (approximately) through the saddle point. We split the exponent of the integrand S as

$$\begin{aligned} S &:= S_1 + S_2, && (1)\\ S_1 &:= \sum_{i=1}^{n} [\ln(1+z^i) + \ln(1+z^{-i}) - \ln 4], \\ S_2 &:= -(j+1) \ln z. \end{aligned}$$

Set

$$S^{(i)} := \frac{d^i S}{dz^i}.$$

As general references for the application of the saddle point method in enumeration we cite [4, 6]. To use this method, we must find the solution of

$$S^{(1)}(\tilde{z}) = 0. \tag{2}$$

Set $\tilde{z} := z^* - \varepsilon$, where, here, $z^* = 1$. (This notation always means that z^* is the approximate saddle point and $\tilde{z}$ is the exact saddle point; they differ by a quantity that has to be computed to some degree of accuracy.) This leads, to first order, to

$$-j - 1 + \Big(-\frac{(n+1)^3}{6} + \frac{(n+1)^2}{4} - \frac{n}{12} - \frac{13}{12} - j\Big)\varepsilon = 0. \tag{3}$$

Set $j = x\sigma$ in (3). This shows that, asymptotically, ε is given by a *Puiseux series* of powers of $n^{-1/2}$, starting with $-\sqrt{6}x/n^{3/2}$. To obtain the next terms, we compute the following terms in the expansion of (2). Even powers ε^{2k} lead to a $\mathcal{O}(n^{2k+1}) \cdot \varepsilon^{2k}$ term and odd powers ε^{2k+1} lead to a $\mathcal{O}(n^{2k+3}) \cdot \varepsilon^{2k+1}$ term. Now we expand into powers of $n^{-1/2}$ and equate each coefficient with 0. This leads successively to a full expansion of ε. Note that to obtain a given precision of ε, it is enough to compute a given finite number of terms in the generalization of (3). We obtain (here and in the following, we provide only the first terms, but we computed more with Maple)

$$\varepsilon = -\sqrt{6}xn^{-3/2} + \sqrt{6}(-3/10x^3 + 3/4x^2)n^{-5/2} + (-3x^2 - 6)n^{-3} + \cdots$$

We have, with $\tilde{z} := z^* - \varepsilon$,

$$P_n(j) = \frac{1}{2\pi i}\int_\Omega \exp\Big[S(\tilde{z}) + S^{(2)}(\tilde{z})(z-\tilde{z})^2/2! + \sum_{l=3}^{\infty} S^{(l)}(\tilde{z})(z-\tilde{z})^l/l!\Big]dz$$

(note carefully that the linear term vanishes). Set $z = \tilde{z} + i\tau$. This gives

$$P_n(j) = \frac{1}{2\pi}\exp[S(\tilde{z})]\int_{-\infty}^{\infty} \exp\Big[S^{(2)}(\tilde{z})(i\tau)^2/2! + \sum_{l=3}^{\infty} S^{(l)}(\tilde{z})(i\tau)^l/l!\Big]d\tau. \tag{4}$$

Let us first analyze $S(\tilde{z})$. We obtain

$$\begin{aligned} S_1(\tilde{z}) &= \frac{x^2}{2} + \frac{9x^4}{40n} + \frac{\sqrt{6}x}{n^{3/2}} + \cdots \\ S_2(\tilde{z}) &= -x^2 - \frac{3x^4}{10n} - \frac{2\sqrt{6}x}{n^{3/2}} + \cdots, \end{aligned}$$

and so

$$S(\tilde{z}) = -\frac{x^2}{2} - \frac{3x^4}{40n} - \frac{\sqrt{6}x}{n^{3/2}} + \cdots$$

Also,

$$\begin{aligned} S^{(2)}(\tilde{z}) &= \frac{n^3}{6} + \Big(-\frac{3}{20}x^2 + \frac{1}{4}\Big)n^2 + \mathcal{O}(n^{3/2}), \\ S^{(3)}(\tilde{z}) &= \mathcal{O}(n^{7/2}), \\ S^{(4)}(\tilde{z}) &= -\frac{1}{20}n^5 + \mathcal{O}(n^4), \\ S^{(l)}(\tilde{z}) &= \mathcal{O}(n^{l+1}), \quad l \geq 5. \end{aligned}$$

We can now compute (4), for instance by using the classical trick of setting

$$S^{(2)}(\tilde{z})(i\tau)^2/2! + \sum_{l=3}^{\infty} S^{(l)}(\tilde{z})(i\tau)^l/l! = -u^2/2, \tag{5}$$

computing τ as a truncated series in u, starting as

$$\tau = \frac{u}{n^{3/2}}\sqrt{6} + \cdots$$

setting $d\tau = \frac{d\tau}{du}du$, expanding w.r.t. n and integrating on $[u = -\infty..\infty]$. (This amounts to the *reversion* of a series.) Finally (4) leads to

$$P_n(j) \sim e^{-x^2/2} \cdot \exp\Big([-39/40 + 9/20x^2 - 3x^4/40]/n + \mathcal{O}(n^{-3/2})\Big)/(2\pi n^3/6)^{1/2}. \tag{6}$$

Note that $S^{(3)}(\tilde{z})$ does not contribute to the $1/n$ correction. Note also that, unlike in the instance of the number of inversions in permutations (see [5]), we have an x^4 term in the first order correction. Let us finally remark that Clark's result is just the main term of (6), with $x = 0$, as he chooses $m = o(\sigma) = o(n^{3/2})$.

To check the effect of the correction, we first give in Figure 1, for $n = 60$, the comparison between $P_n(j)$ and the asymptotics (6), without the $1/n$ term (i.e. the Gaussian contribution). To get a better quality estimation, Figure 2 shows $P_n(j)$ and the asymptotics (6), with the constant term $-39/(40n)$. Figure 3 shows the quotient Q_1 of $P_n(j)$ and the asymptotics (6), without the $1/n$ term. The usual "hat" behaviour is apparent.

Figure 4 shows Q_1 and Q_2: the quotient of $P_n(j)$ and the asymptotics (6), with the constant term $-39/(40n)$.

Figure 5 shows Q_2 and Q_3: the quotient of $P_n(j)$ and the asymptotics (6), with the constant term $-39/(40n)$ and the x^2 term $9x^2/(20n)$.

Finally, Figure 6 shows Q_3 and Q_4: the quotient of $P_n(j)$ and the full asymptotics (6). Q_4 gives indeed a good precision on a larger range.

For the reader's convenience, here are the quantities that we plot:

$$Q_1 = P_n(j) \Big/ \left[e^{-x^2/2}/(2\pi n^3/6)^{1/2}\right],$$

$$Q_2 = P_n(j) \Big/ \left[e^{-x^2/2} \cdot \exp[(-39/40)/n]/(2\pi n^3/6)^{1/2}\right],$$

$$Q_3 = P_n(j) \Big/ \left[e^{-x^2/2} \cdot \exp[(-39/40 + 9/20x^2)/n]/(2\pi n^3/6)^{1/2}\right],$$

$$Q_4 = P_n(j) \Big/ \left[e^{-x^2/2} \cdot \exp[(-39/40 + 9/20x^2 - 3x^4/40)/n]/(2\pi n^3/6)^{1/2}\right].$$

3 The case $j = n^{7/4} - x$

Now we show that our methods are strong enough to deal with probabilities that are far away from the average; viz. $n^{7/4} - x$, for fixed x. Of course, they are very small, but nevertheless

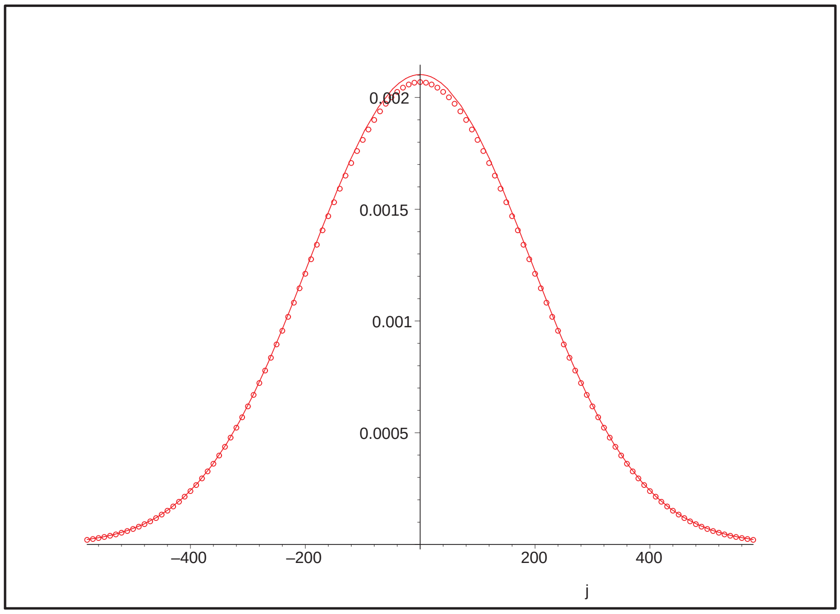

Figure 1: $P_n(j)$ (circle) and the Gaussian asymptotics (line), $n = 60$

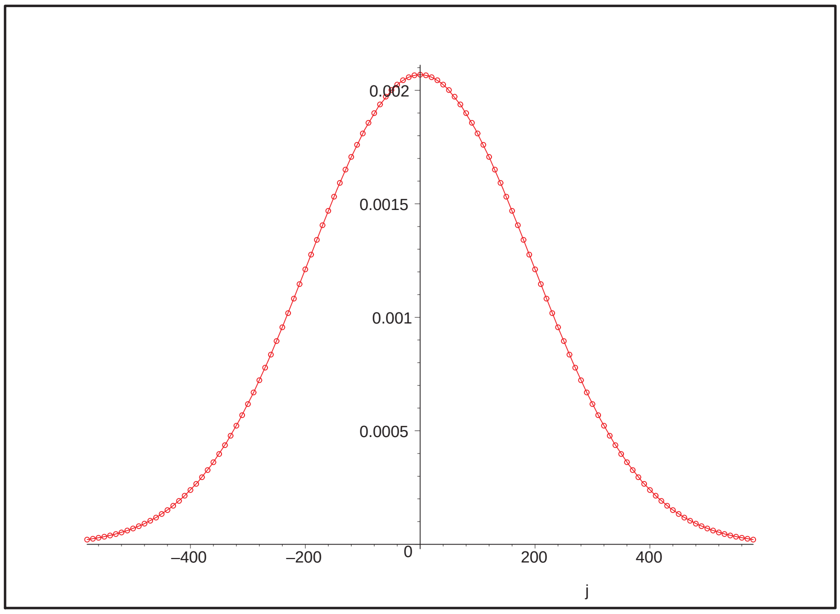

Figure 2: $P_n(j)$ (circle) and the Gaussian asymptotics with the constant term (line), $n = 60$

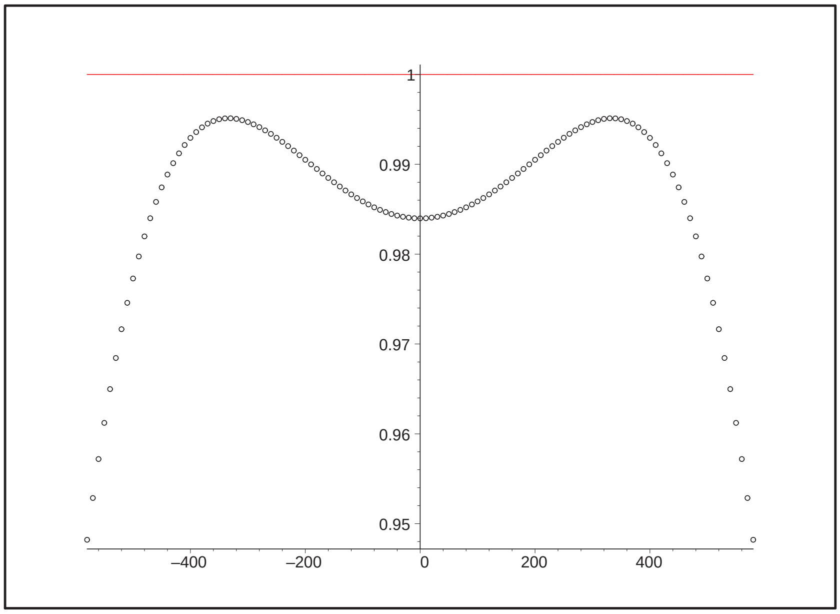

Figure 3: Q_1

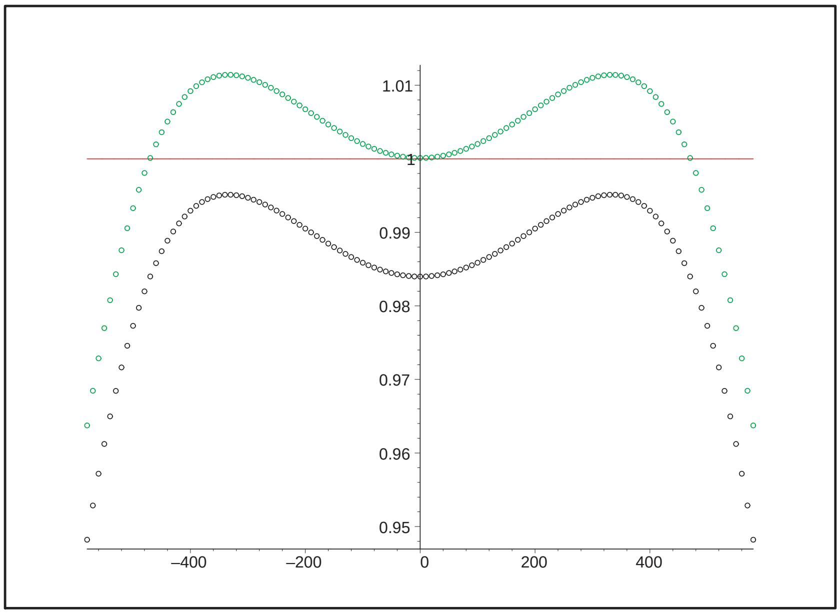

Figure 4: Q_1 (black) and Q_2 (green)

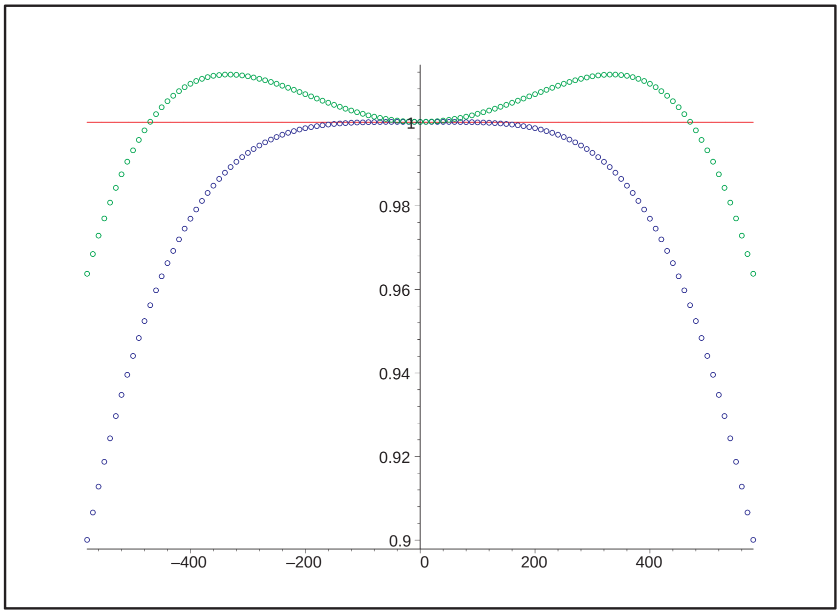

Figure 5: Q_2 (green) and Q_3 (blue)

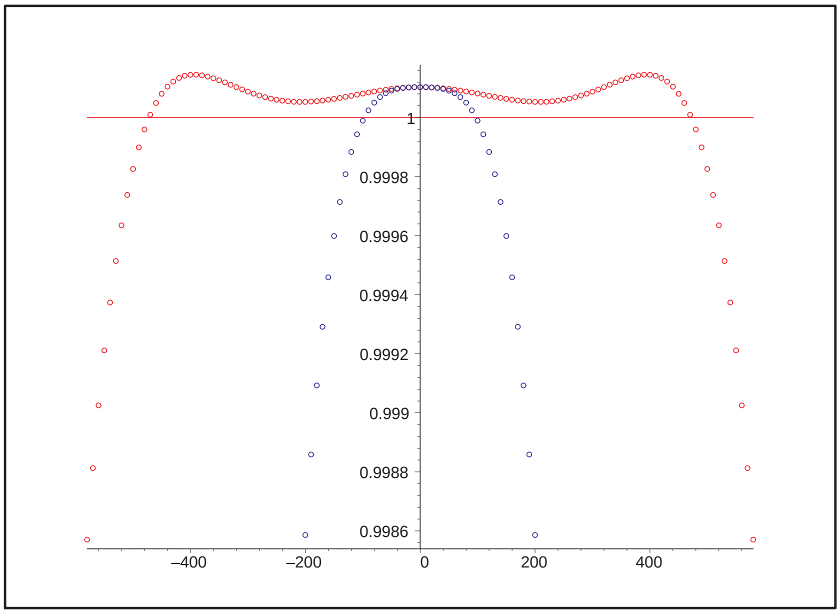

Figure 6: Q_3 (blue) and Q_4 (red)

we find asymptotic formulæ for them. Later on, $n^{7/4}-x$ will be used as an integer. Equation (3) becomes now

$$-n^{7/4}+x-1+\Big(-\frac{(n+1)^3}{6}+\frac{(n+1)^2}{4}-\frac{n}{12}-\frac{13}{12}+x\Big)\varepsilon=0.$$

We have $z^*=1$ and, as we will see, ε is now given by a *Puiseux series* of powers of $n^{-1/4}$. We obtain

$$\varepsilon=-6n^{-5/4}-\frac{54}{5}n^{-7/4}-\frac{3771}{175}n^{-9/4}+\cdots$$

Again we provide only a few terms, but we mention that more terms than in the previous section are usually necessary. This leads to

$$\begin{aligned}
S_1(\tilde z)&=3\sqrt{n}+\frac{81}{10}+\frac{1467}{70\sqrt{n}}+\cdots,\\
S_2(\tilde z)&=-6\sqrt{n}-\frac{54}{5}-\frac{3771}{175\sqrt{n}}+\cdots,
\end{aligned}$$

and so

$$S(\tilde z)=-3\sqrt{n}-\frac{27}{10}-\frac{207}{350\sqrt{n}}+\cdots.$$

Let us mention that x appears only in the $n^{-5/4}$ term and x^2 appears (first) in the n^{-3} term. Also,

$$\begin{aligned}
S^{(2)}(\tilde z)&=\frac{1}{6}n^3-\frac{9}{10}n^{5/2}+\frac{607}{700}n^2+\cdots,\\
S^{(3)}(\tilde z)&-\frac{3}{10}n^{15/4}+\frac{711}{350}n^{13/4}+\cdots,\\
S^{(4)}(\tilde z)&=-\frac{1}{20}n^5+\cdots,\\
S^{(5)}(\tilde z)&=\mathcal{O}(n^{23/4}).
\end{aligned}$$

This finally gives

$$\begin{aligned}
P_n(j)\sim{}&e^{-3n^{1/2}-27/10}\Big[1+369/175/n^{1/2}+931359/245000/n+1256042967/471625000/n^{3/2}\\
&+4104x/175/n^{7/4}-9521495156603/2145893750000/n^2+7561917x/122500/n^{9/4}+\\
&(-235974412468129059/341392187500000+18x^2)/n^{5/2}+\cdots\Big]\Big/(2\pi n^3/6)^{1/2}. \qquad (7)
\end{aligned}$$

Figure 7 shows, for $n=150$, $P_n(j)$ and the full asymptotics (7) up to the $n^{-5/2}$ term, both scaled to 10^{21}. The fit is quite satisfactory.

References

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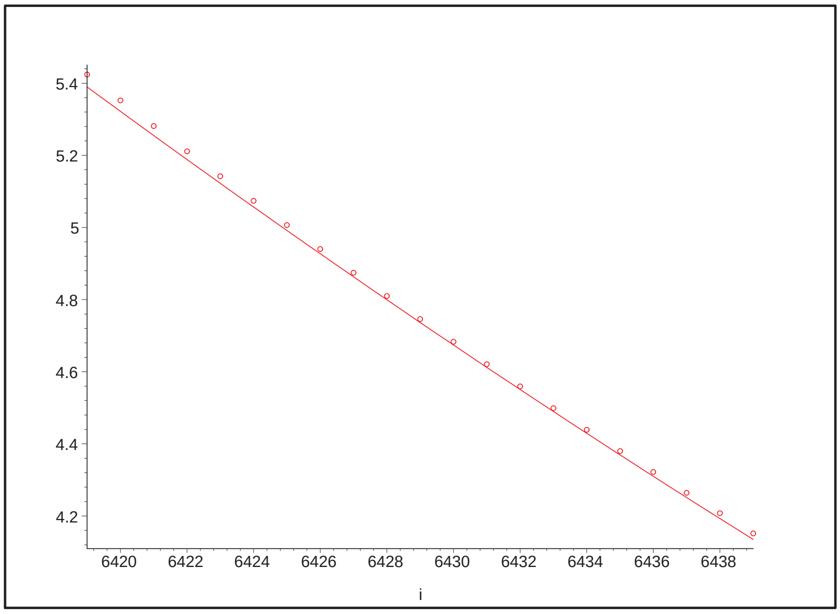

Figure 7: $P_n(j)$ (circle) and the full asymptotics (7) (line), $n = 150$, scaling$= 10^{21}$

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