Number of Singletons in Involutions of large size: a central range and a large deviation analysis

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Abstract

In this paper, we analyze the asymptotic number $I(m,n)$ of involutions of large size n with m singletons. We consider a central region and a non-central region. In the range $m = n - n^{\alpha}, 0 < \alpha < 1$, we analyze the dependence of $I(m,n)$ on α. This paper fits within the framework of Analytic Combinatorics.

Keywords: Involutions, Singletons, Asymptotics, Saddle point method, Multiseries expansions, Analytic Combinatorics.

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1 Introduction

During the last few years, we have been interested in asymptotic properties of some permutations parameters. For instance, in [9] (in cooperation with H.Prodinger), [7], [8], we analyzed the number of inversions, of cycles (related to the Stirling numbers of the first kind), of rises (related to Eulerian numbers). We extended the Gaussian approximation with more terms and we also considered some large deviation expansions. In this paper, we turn to another property: the number of singletons in involutions: an involution is a permutation σ such that σ^2 is the identity permutation. this corresponds to cycles of size 1 and 2. See Bona [1] for details. We will use the Saddle point method: see Flajolet and Sedgewick [2, ch. VIII] for a nice introduction.

We denote by I_n the total number of involutions of size n and by $I(m,n)$ the number of all involutions of size n with m singletons . We define a random variable J_n by the relation

$$\mathbb{P}(J_n = m) = \frac{I(m,n)}{I_n}.$$

This the number of singletons in an involution chosen (uniformly) at random among all involutions of size n. In [2, ch. VIII, p.558], using the Saddle point technique, Flajolet and Sedgewick give the first terms of the asymptotic expansion of I_n, also obtained by Knuth [6]. See also Moser and Wyman [10]. In Section 2, we provide a more detailed expansion of I_n. In [2, ch. VIII, p.691], the authors provide the first terms of the mean and variance of J_n. In Section 2, we consider a detailed analysis of all moments of J_n. In [2, ch. VIII, p.692], the authors prove the dominant Gaussian asymptotic of J_n by using together the Saddle point technique and a generalized quasi-powers technique (see Sachkov [11], Hwang [4], [5]). In Section 3, we give a detailed analyzis of the asymptotic distribution of J_n. In Section 4, we consider a large deviation range: $m = n - n^{\alpha}, 0 < \alpha < 1$. In this section, we will use multiseries expansions: multiseries are in effect power series (in which the powers may be non-integral but must tend to infinity) and the variables are elements of a scale. The scale is a set of variables of increasing order. The series is computed in terms of the variable of maximum order, the coefficients

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of which are given in terms of the next-to-maximum order, etc. This is more precise than mixing different terms.

An appendix provides a justification of some integration procedures.

Note finally that our approach can be used in generalizations of the involution: we can deal with cycles of any chosen sizes and deal with singletons or other specific cycle.

2 The moments

We have the classical (exponential) generating functions

$$\begin{aligned} f_1(z) &= \sum_{n=0}^{\infty} I_n \frac{z^n}{n!} = e^{z+z^2/2}, \\ f_2(z,y) &= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} I(m,n)\frac{z^n}{n!}y^m = e^{zy+z^2/2}, \\ f_3(z,m) &= \sum_{n=0}^{\infty} I(m,n)\frac{z^n}{n!} = e^{z^2/2}\frac{z^m}{m!}. \end{aligned}$$

We note that m and n do have the same parity: $n-m$ is even. From $f_3(z,m)$, we have

$$\frac{I(m,n)}{n!} = \frac{1}{2^{(n-m)/2}((n-m)/2)!m!}. \tag{1}$$

We define $m^{\underline{\ell}} := \prod_{j=0}^{\ell-1}(m-j)$ as the ℓth falling factorial of m.

We have

$$\begin{aligned} \mathbb{E}(J_n^{\underline{\ell}}) &= \sum_{m=0}^{\infty} m^{\underline{\ell}}\mathbb{P}(J_n=m) = \frac{S_l}{I_n}, \\ S_l &= \sum_{m=0}^{\infty} m^{\underline{\ell}} I(m,n) = n!\,[z^n]\,\frac{\partial^\ell}{\partial y^\ell} f_2(z,y)\Big|_{y=1} = n!\,[z^n]z^\ell e^{z+z^2/2} = n![z^{n-\ell}]e^{z+z^2/2}, \\ \mathbb{E}(J_n^{\underline{\ell}}) &= \frac{n!}{I_n}\frac{I_{n-\ell}}{(n-\ell)!}. \end{aligned}$$

Now we turn to an asymptotic expansion of I_n.

Let Ω denote the circle $\rho e^{\mathbf{i}\theta}$. By Cauchy's theorem, it follows that

$$\begin{aligned} I_n/n! &= \frac{1}{2\pi\mathbf{i}}\int_\Omega \frac{f_1(z)}{z^{n+1}}\,dz \\ &= \frac{1}{\rho^n}\frac{1}{2\pi}\int_{-\pi}^{\pi} f_1(\rho e^{\mathbf{i}\theta})e^{-n\mathbf{i}\theta}\,d\theta \qquad \text{using } z=\rho e^{\mathbf{i}\theta} \\ &= \frac{1}{\rho^n}\frac{1}{2\pi}\int_{-\pi}^{\pi} \exp\left(\ln(f_1(\rho e^{\mathbf{i}\theta})) - n\mathbf{i}\theta\right)d\theta \\ &= \frac{1}{\rho^n}\frac{f_1(\rho)}{2\pi}\int_{-\pi}^{\pi}\exp\left[\mathbf{i}\kappa_1\theta - n\mathbf{i}\theta - \frac{1}{2}\kappa_2\theta^2 - \frac{\mathbf{i}}{6}\kappa_3\theta^3 + \cdots\right]d\theta, \end{aligned} \tag{2}$$

where

$$\kappa_i(\rho) := \left(\frac{\partial}{\partial u}\right)^i \ln(f_1(\rho e^u))\big|_{u=0}.$$

Now we have $\kappa_1 = \rho+\rho^2$ and we set $\kappa_1 - n = 0$ such that the saddle point is the root (of smallest modulus) of $\rho+\rho^2-n=0$ (From now on, we only provide a few terms in our expansions, but of course we use more terms in our computations). This leads to

$$
\begin{aligned}
\rho &= -\frac{1}{2} + \frac{1}{2}\sqrt{4n+1} \\
&= \sqrt{n} - 1/2 + 1/8\,\frac{1}{\sqrt{n}} - \frac{1}{128}\,n^{-3/2} + \frac{1}{1024}\,n^{-5/2} - \frac{5}{32768}\,n^{-7/2} + \mathcal{O}\left(\frac{1}{n^{9/2}}\right), \\
\ln(\rho) &= 1/2\,\ln(n) - 1/2\,\frac{1}{\sqrt{n}} + 1/48\,n^{-3/2} - \frac{3}{1280}\,n^{-5/2} + \frac{5}{14336}\,n^{-7/2} - \frac{35}{589824}\,n^{-9/2} + \mathcal{O}\left(\frac{1}{n^{5}}\right).
\end{aligned}
$$

See Good [3] for a neat description of this technique.

The dominant part of (2) gives

$$
\frac{f_1(\rho)}{\rho^n} = \exp(E_1),
$$
$$
E_1 = \rho + \rho^2/2 - n\ln(\rho) = n/2 + \rho/2 - n\ln(\rho),
$$

with the substitution $\rho^2 = n - \rho$. (This substitution will be frequently used in the sequel.)

Now we turn to the integral. We have

$$
\kappa_2 = -\rho + 2n,
$$

and more generally

$$
\kappa_j = -(2^{j-1}-1)\rho + 2^{j-1}n.
$$

We choose a splitting value θ_0 such that $\kappa_2\theta_0^2 \to \infty$, and $\kappa_3\theta_0^3 \to 0$, as $n \to \infty$. If we choose $\theta_0 = n^{\beta}$, we must have $n^{2\beta+1} \to \infty, n^{3\beta+1} \to 0$. For instance, we can use $\theta_0 = n^{-5/12}$. We must prove that the integral

$$
K_n = \int_{\theta_0}^{2\pi-\theta_0} \exp\left(\ln(f_1(\rho e^{\mathbf{i}\theta})) - n\mathbf{i}\theta\right) d\theta
$$

is such that $|K_n|$ is exponentially small (tail pruning). This is done in [2, ch. VIII, p.559]. Now we use the classical trick of setting

$$
\sum_{j=2}^{\infty} \kappa_j(\mathbf{i}\theta)^j/j! = \frac{1}{2}\left[(n-\rho)(e^{2\mathbf{i}\theta} - 1 - 2\mathbf{i}\theta)\right] + \rho(e^{\mathbf{i}\theta} - 1 - \mathbf{i}\theta) = -u^2/2.
$$

Computing θ as a series in u, this gives, by Lagrange's inversion,

$$
\theta = \sum_{i=1}^{\infty} a_i \frac{u^i}{n^{i/2}}, \tag{3}
$$

with, for instance,

$$
\begin{aligned}
a_1 &= 1/2\,\sqrt{2} + 1/8\,\frac{\sqrt{2}}{\sqrt{n}} - \frac{1}{64}\,\frac{\sqrt{2}}{n} - \frac{3}{256}\,\frac{\sqrt{2}}{n^{3/2}} + \frac{11}{4096}\,\frac{\sqrt{2}}{n^2} + \mathcal{O}\left(\frac{1}{n^{5/2}}\right), \\
a_2 &= -1/6\,i - \frac{1/24\,i}{\sqrt{n}} + \frac{1/48\,i}{n} + \frac{\frac{1}{192}\,i}{n^{3/2}} - \frac{\frac{1}{192}\,i}{n^2} + \mathcal{O}\left(\frac{1}{n^{5/2}}\right).
\end{aligned}
$$

This expansion is valid in the dominant integration domain

$$
|u| \le \frac{\sqrt{n}\,\theta_0}{a_1} = n^{1/12}.
$$

Setting $d\theta = \frac{d\theta}{du}du$, we integrate on $u = [-\infty, \infty]$. The extension of the range (tail completion) is justified in [2, ch. VIII, p.562]. The same justification is applicable in the next sections. The integration gives

$$\frac{1}{2\sqrt{\pi}\sqrt{n}}F_1,$$

$$F_1 = 1 + 1/4\,\frac{1}{\sqrt{n}} - \frac{19}{96}\,n^{-1} - \frac{13}{384}\,n^{-3/2} + \mathcal{O}\left(\frac{1}{n^2}\right),$$

$$\frac{I_n}{n!} \sim \frac{1}{2\sqrt{\pi}\sqrt{n}}F_1\exp(E_1).$$

Now we turn to $\frac{I_{n-\ell}}{(n-\ell)!}$. We successively have

$$\rho_\ell + \rho_\ell^2 - n + \ell = 0,$$

$$\rho_\ell^2 = n - \ell - \rho_\ell \text{ is used as a next substitution,}$$

$$\rho_\ell = \sqrt{n} - 1/2 + \frac{1/8 - 1/2\,\ell}{\sqrt{n}} + \left(-\frac{1}{128} + 1/16\,\ell - 1/8\,\ell^2\right)n^{-3/2} + \left(\frac{1}{1024} - \frac{3}{256}\,\ell + \frac{3}{64}\,\ell^2 - 1/16\,\ell^3\right)n^{-5/2}$$
$$+ \left(-\frac{5}{32768} + \frac{5}{2048}\,\ell - \frac{15}{1024}\,\ell^2 + \frac{5}{128}\,\ell^3 - \frac{5}{128}\,\ell^4\right)n^{-7/2} + \mathcal{O}\left(\frac{1}{n^{9/2}}\right),$$

$$\ln(\rho_\ell) = 1/2\,\ln(n) - 1/2\,\frac{1}{\sqrt{n}} - 1/2\,\frac{\ell}{n} + \frac{1/48 - 1/4\,\ell}{n^{3/2}} - 1/4\,\frac{\ell^2}{n^2} + \left(-\frac{3}{1280} + 1/32\,\ell - 3/16\,\ell^2\right)n^{-5/2} - 1/6\,\frac{\ell^3}{n^3}$$
$$+ \left(\frac{5}{14336} - \frac{3}{512}\,\ell + \frac{5}{128}\,\ell^2 - \frac{5}{32}\,\ell^3\right)n^{-7/2} - 1/8\,\frac{\ell^4}{n^4}$$
$$+ \left(-\frac{35}{589824} + \frac{5}{4096}\,\ell - \frac{21}{2048}\,\ell^2 + \frac{35}{768}\,\ell^3 - \frac{35}{256}\,\ell^4\right)n^{-9/2} + \mathcal{O}\left(\frac{1}{n^5}\right),$$

$$\frac{f_1(\rho_\ell)}{\rho_\ell^n} = \exp(E_{1,\ell}),$$

$$E_{1,\ell} = n/2 + \rho_\ell/2 - \ell/2 - (n-\ell)\ln(\rho_\ell),$$

$$\kappa_{j,\ell} = -(2^{j-1}-1)\rho_\ell + 2^{j-1}(n-\ell),$$

$$\theta \text{ is again given by (3),}$$

$$a_{1,\ell} = 1/2\,\sqrt{2} + 1/8\,\frac{\sqrt{2}}{\sqrt{n}} + \left(-\frac{1}{64}\,\sqrt{2} + 1/4\,\ell\sqrt{2}\right)n^{-1} + \left(1/8\,\ell\sqrt{2} - \frac{3}{256}\,\sqrt{2}\right)n^{-3/2} + \mathcal{O}\left(\frac{1}{n^2}\right),$$

$$F_{1,\ell} = 1 + 1/4\,\frac{1}{\sqrt{n}} + \left(-\frac{19}{96} + 1/2\,\ell\right)n^{-1} + \left(1/4\,\ell - \frac{13}{384}\right)n^{-3/2} + \mathcal{O}\left(\frac{1}{n^2}\right),$$

$$\frac{I_{n-\ell}}{(n-\ell)!} \sim \frac{1}{2\sqrt{\pi}\sqrt{n}}F_{1,\ell}\exp(E_{1,\ell}).$$

Note that setting $\ell = 0$, we recover of course I_n. The detailed expansions of ρ_ℓ and $\ln(\rho_\ell)$ are used in E_1 and $E_{1,\ell}$.

We are now ready to compute $\mathbb{E}(J_n^{\underline{\ell}}) = \frac{S_\ell}{I_n}$. We derive

$$\exp(E_{1,\ell} - E_1) = n^{\ell/2} T_1,$$

$$T_1 = 1 - 1/2\,\frac{\ell}{\sqrt{n}} - 1/8\,\frac{\ell^2}{n} + \left(1/48\,\ell - 1/8\,\ell^2 + \frac{5}{48}\,\ell^3\right) n^{-3/2} + \left(-\frac{1}{96}\,\ell^2 + 1/16\,\ell^3 + \frac{1}{384}\,\ell^4\right) n^{-2}$$
$$+ \left(-\frac{1}{384}\,\ell^3 + \frac{1}{64}\,\ell^4 - \frac{41}{3840}\,\ell^5\right) n^{-5/2} + \mathcal{O}\left(\frac{1}{n^3}\right),$$

$$\frac{F_{1,\ell}}{F_1} = T_2,$$

$$T_2 = 1 + 1/2\,\frac{\ell}{n} + 1/8\,\frac{\ell}{n^{3/2}} + \mathcal{O}\left(\frac{1}{n^2}\right),$$

$$T_3 = T_1 T_2 = 1 - 1/2\,\frac{\ell}{\sqrt{n}} + \frac{-1/8\,\ell^2 + 1/2\,\ell}{n} + \left(\frac{7}{48}\,\ell - 3/8\,\ell^2 + \frac{5}{48}\,\ell^3\right) n^{-3/2} + \mathcal{O}\left(\frac{1}{n^2}\right).$$

This leads to the following theorem:

Theorem 2.1 *The asymptotic expansion of the factorial moments of J_n are given by*

$$\mathbb{E}(J_n^{\underline{\ell}}) = n^{\ell/2}\left[1 - 1/2\,\frac{\ell}{\sqrt{n}} + \frac{-1/8\,\ell^2 + 1/2\,\ell}{n} + \left(\frac{7}{48}\,\ell - 3/8\,\ell^2 + \frac{5}{48}\,\ell^3\right) n^{-3/2} + \mathcal{O}\left(\frac{1}{n^2}\right)\right].$$

The first moments of J_n are now immediate:

$$M = \frac{S_1}{I_n} = \sqrt{n} - 1/2 + 3/8\,\frac{1}{\sqrt{n}} - 1/8\,n^{-1} + \mathcal{O}\left(\frac{1}{n^{3/2}}\right),$$

$$\mathbb{E}(J_n(J_n - 1)) = \frac{S_2}{I_n},$$

$$\sigma^2 = \frac{S_2}{I_n} + M - M^2 = \sqrt{n} - 1 + 5/8\,\frac{1}{\sqrt{n}} + \mathcal{O}\left(\frac{1}{n}\right),$$

$$\sigma = \sqrt[4]{n} - 1/2\,\frac{1}{\sqrt[4]{n}} + 3/16\,n^{-3/4} + \mathcal{O}\left(\frac{1}{n}\right).$$

All moments can be similarly mechanically obtained

3 Distribution of J_n

We consider the central range $M - 2\sigma < m < M + 2\sigma$. We have $m = M + x\sigma, x = \Theta(1)$. We first analyze $\frac{I(m,n)}{n!}$.

3.1 Approach 1: Asymptotic expansion of $\frac{I(m,n)}{n!}$ using the Saddle point technique

We derive

$$\kappa_i(\rho) := \left(\frac{\partial}{\partial u}\right)^i \ln(f_3(\rho e^u))|_{u=0},$$

$$\kappa_1 = \rho^2 + m,$$

$$\rho^2 = n - m \text{ (we use that as a substitution in the sequel)},$$

$$\kappa_j = (n - m)2^{j-1},$$

$$\rho = \sqrt{n} - 1/2 - 1/2\,\frac{x}{\sqrt[4]{n}} + 1/8\,\frac{1}{\sqrt{n}} + \frac{-1/8 - 1/8\,x^2}{n} - 1/32\,\frac{x}{n^{5/4}} + \mathcal{O}\left(\frac{1}{n^{3/2}}\right),$$

$$\ln(\rho) = 1/2\,\ln(n) - 1/2\,\frac{1}{\sqrt{n}} - 1/2\,\frac{x}{n^{3/4}} - 1/4\,\frac{x}{n^{5/4}} + \left(-\frac{5}{48} - 1/4\,x^2\right) n^{-3/2} - \frac{3}{32}\,\frac{x}{n^{7/4}} + \mathcal{O}\left(\frac{1}{n^2}\right).$$

The dominant part of (2) gives

$$\frac{f_3(\rho)}{\rho^n} = \exp(E_2), \tag{4}$$

$$E_2 = \rho^2/2 + m\ln(\rho) - \ln(m!) - n\ln(\rho), \tag{5}$$

we know that

$$\ln(\ell!) = -\ell + \ell\ln(\ell) + \frac{1}{2}\ln(2\pi\ell) + \frac{1}{12}\ell^{-1} + \mathcal{O}\left(\frac{1}{\ell^2}\right), \tag{6}$$

hence

$$E_2 = (-1/2\,\ln(n) + 1/2)\,n + \sqrt{n} - 1/4 - 1/2\,x^2 - 1/4\,\ln(n) - 1/2\,\ln(2) - 1/2\,\ln(\pi) + \frac{-1/2\,x + 1/6\,x^3}{\sqrt[4]{n}}$$
$$+ \left(\frac{5}{24} + 1/4\,x^2 - 1/12\,x^4\right)\frac{1}{\sqrt{n}} + \frac{-1/6\,x - 1/4\,x^3 + 1/20\,x^5}{n^{3/4}} + \mathcal{O}\left(\frac{1}{n}\right).$$

Now we turn to the integral. Proceeding as previously, we have

$$\frac{1}{2}(n-m)(e^{2\mathbf{i}\theta} - 1 - 2\mathbf{i}\theta) = -\frac{u^2}{2},$$

θ is again given by (3),

$$a_1 = 1/2\,\sqrt{2} + 1/4\,\frac{\sqrt{2}}{\sqrt{n}} + 1/4\,\frac{\sqrt{2}x}{n^{3/4}} + 1/16\,\frac{\sqrt{2}}{n} + 1/4\,\frac{\sqrt{2}x}{n^{5/4}}$$
$$+ \left(-1/16\,\sqrt{2} + 1/2\,\sqrt{2}\left(-1/8 - 1/8\,x^2\right) - 1/4\,x^2\sqrt{2} - 8\left(-1/16\,x^2 - \frac{3}{128}\right)\sqrt{2}\right)n^{-3/2} + \frac{7}{64}\,\frac{\sqrt{2}x}{n^{7/4}} + \mathcal{O}\left(\frac{1}{n^2}\right)$$

$$a_2 = -1/6\,\mathbf{i} - \frac{1/6\,\mathbf{i}}{\sqrt{n}} - \frac{1/6\,\mathbf{i}x}{n^{3/4}} - \frac{1/12\,\mathbf{i}}{n} - \frac{1/4\,\mathbf{i}x}{n^{5/4}} + \frac{-1/6\,\mathbf{i}x^2 - 1/16\,\mathbf{i}}{n^{3/2}} - \frac{\frac{17}{96}\,\mathbf{i}x}{n^{7/4}} + \mathcal{O}\left(\frac{1}{n^2}\right)$$

This integration gives

$$\frac{1}{2\sqrt{\pi}\sqrt{n}}F_2,$$

$$F_2 = 1 + 1/2\,\frac{1}{\sqrt{n}} + 1/2\,\frac{x}{n^{3/4}} - 1/24\,n^{-1} + 1/2\,\frac{x}{n^{5/4}} \tag{7}$$

$$+ \frac{1}{96}\,\frac{-12\,\sqrt{\pi} + 36\,\sqrt{\pi}x^2}{\sqrt{\pi}n^{3/2}} - 1/32\,\frac{x}{n^{7/4}} + \mathcal{O}\left(\frac{1}{n^2}\right), \tag{8}$$

$$\frac{I(m,n)}{n!} \sim \frac{1}{2\sqrt{\pi}\sqrt{n}}F_2\exp(E_2). \tag{9}$$

3.2 Approach 2: Asymptotic expansion of $\frac{I(m,n)}{n!}$ using its explicit expression

We use again $\ln(\ell!)$ as given by (6). We successively obtain

$$\begin{aligned}
m &= \sqrt{n} + x\sqrt[4]{n} - 1/2 - 1/2\,\frac{x}{\sqrt[4]{n}} + 3/8\,\frac{1}{\sqrt{n}} + 3/16\,\frac{x}{n^{3/4}} + \mathcal{O}(\frac{1}{n}),\\
(n-m)/2 &= 1/2\,n - 1/2\,\sqrt{n} - 1/2\,x\sqrt[4]{n} + 1/4 + 1/4\,\frac{x}{\sqrt[4]{n}} - 3/16\,\frac{1}{\sqrt{n}} - \frac{3}{32}\,\frac{x}{n^{3/4}} + \mathcal{O}(\frac{1}{n}),\\
\ln(m!) &= \left(-1+\ln\left(\sqrt{n}\right)\right)\sqrt{n} + x\ln\left(\sqrt{n}\right)\sqrt[4]{n} + 1/2\,x^2 + 1/2\,\ln(2) + 1/2\,\ln(\pi)\\
&+ \frac{-1/6\,x^3 - 1/2\,x\ln\left(\sqrt{n}\right)}{\sqrt[4]{n}} + \frac{-1/24 - 1/2\,x^2 + 1/12\,x^4 + 3/8\,\ln\left(\sqrt{n}\right)}{\sqrt{n}}\\
&+ \left(\frac{5}{12}\,x - 1/20\,x^5 + 3/16\,x\ln\left(\sqrt{n}\right) + 1/4\,x^3\right)n^{-3/4} + \mathcal{O}(\frac{1}{n}),\\
\ln(((n-m)/2)!) &= \left(-1/2 - 1/2\,\ln(2) + \ln\left(\sqrt{n}\right)\right)n + \left(1/2\,\ln(2) - \ln\left(\sqrt{n}\right)\right)\sqrt{n} + \left(1/2\,x\ln(2) - x\ln\left(\sqrt{n}\right)\right)\sqrt[4]{n}\\
&+ 1/4 - 1/4\,\ln(2) + 3/2\,\ln\left(\sqrt{n}\right) + 1/2\,\ln(\pi) + \frac{1/2\,x - 1/4\,x\ln(2) + 1/2\,x\ln\left(\sqrt{n}\right)}{\sqrt[4]{n}} + \mathcal{O}(\frac{1}{\sqrt{n}}).
\end{aligned}$$

This leads to

$$\frac{\tilde{I}(m,n)}{n!} = \exp[(-1/2\,\ln(n) + 1/2)\,n + \sqrt{n} - 1/2\,\ln(2) - 1/4 - 1/2\,x^2 - 3/4\,\ln(n) - \ln(\pi) + \frac{\left(1/6\,x^3 - 1/2\,x\right)}{\sqrt[4]{n}} + \mathcal{O}(\frac{1}{\sqrt{n}})].$$

We observe two peculiarities: If we compare $\frac{\tilde{I}(m,n)}{n!}$ with $\frac{I(m,n)}{n!}$ as given by (9), We observe, after some algebra, two peculiarities:

- $\frac{\tilde{I}(m,n)}{n!} \sim 2\frac{I(m,n)}{n!}$. This will be explained in the sequel.
- much more initial asymptotic precision is needed in Approach 2 in order to obtain the same final precision in the coefficient ($\mathcal{O}(\frac{1}{n^2})$) as in Approach 1. This is due to the fact that we use two $\ln(\ell!)$ asymptotics in Approach 2 instead of one in Approach 2.

3.3 Distribution of J_n

So, finally

$$\begin{aligned}
\frac{I(m,n)}{I_n} &= F_2/F_1\exp(E_2 - E_1),\\
\exp(E_2 - E_1) &= \frac{1}{\sqrt{2\pi}n^{1/4}}e^{-x^2/2}F_3,\\
F_3 &= 1 + \frac{-1/2\,x + 1/6\,x^3}{\sqrt[4]{n}} + \left(1/6 + 3/8\,x^2 - 1/6\,x^4 + \frac{1}{72}\,x^6\right)\frac{1}{\sqrt{n}}\\
&+ \left(-1/4\,x - \frac{53}{144}\,x^3 + \frac{37}{240}\,x^5 - 1/48\,x^7 + \frac{1}{1296}\,x^9\right)n^{-3/4} + \mathcal{O}\left(\frac{1}{n}\right),\\
\frac{I(m,n)}{I_n} &= \frac{1}{\sqrt{2\pi}n^{1/4}}e^{-x^2/2}F_4,\\
F_4 &= F_2/F_1F_3 = 1 + \frac{-1/2\,x + 1/6\,x^3}{\sqrt[4]{n}} + \left(\frac{5}{12} + 3/8\,x^2 - 1/6\,x^4 + \frac{1}{72}\,x^6\right)\frac{1}{\sqrt{n}}\\
&+ \left(1/8\,x - \frac{47}{144}\,x^3 + \frac{37}{240}\,x^5 - 1/48\,x^7 + \frac{1}{1296}\,x^9\right)n^{-3/4} + \mathcal{O}\left(\frac{1}{n}\right).
\end{aligned}$$

This leads to the local limit theorem:

Theorem 3.1 *The asymptotic distribution of J_n in the central range is given by the local limit theorem:*

$$\mathbb{P}(J_n = m) = 2\frac{1}{\sqrt{2\pi}n^{1/4}}e^{-x^2/2}\left[1 + \frac{-1/2\,x + 1/6\,x^3}{\sqrt[4]{n}} + \left(\frac{5}{12} + 3/8\,x^2 - 1/6\,x^4 + \frac{1}{72}\,x^6\right)\frac{1}{\sqrt{n}} + \left(1/8\,x - \frac{47}{144}\,x^3 + \frac{37}{240}\,x^5 - 1/48\,x^7 + \frac{1}{1296}\,x^9\right)n^{-3/4} + \mathcal{O}\left(\frac{1}{n}\right)\right]. \quad (10)$$

Of course more terms can be mechanically computed, but the expressions become much more intricate. Note carefully the factor 2 in front of our expression: this is justified in the Appendix. This is also justified by probabilistic reasoning: as only one over 2 values of m leads to a non-zero $I(m,n)$ expression, the probability must be multiplied by 2. The tail pruning is also considered in the Appendix: the choice of θ_0 is the same as for I_n.

To check the quality of our asymptotics, we have chosen $n = 2000$. this gives $M = 44.22968229\ldots, \sigma = 6.613262555\ldots$, a range $m \in [30, 58]$.

Figure 1 shows $I(m,n)/I_n$ (circle) and a first asymptotic $2\frac{e^{-x^2/2}}{\sqrt{2\pi}\sigma}$ (line).

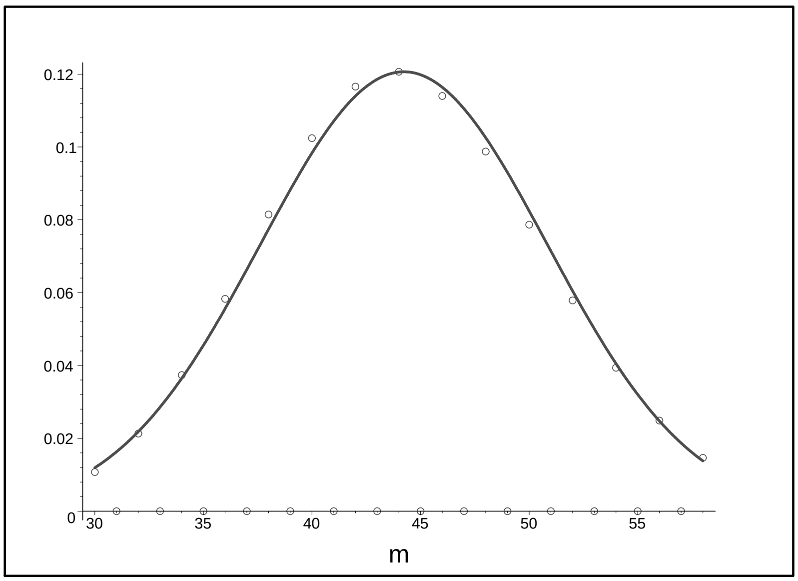

Figure 1: $I(m,n)/I_n$ (circle) and a first asymptotic $2\frac{e^{-x^2/2}}{\sqrt{2\pi}\sigma}$ (line)

Figure 2 shows $I(m,n)/I_n$ (circle) and the asymptotic Equ. (10) (line). The fit is better.

Figure 3 gives the quotient of $I(m,n)/I_n$ and the asymptotic $2\frac{e^{-x^2/2}}{\sqrt{2\pi}\sigma}$ (box) as well as the quotient of $I(m,n)/I_n$ and the asymptotic Equ. (10) (circle).

4 Large deviation $m = n - n^\alpha, 0 < \alpha < 1$

$lnmft = -n + nL + 1/2\,n2alm1 + 1/6\,n3alm2 + 1/12\,n4alm3 - nal\,L + 1/4\,n5alm4 + 1/2\,\ln(2\,\pi\,n\,(1 - eps)) + 1/12\,\frac{1}{n(1-eps)} - \frac{1}{360}\,\frac{1}{n^3(1-eps)^3} + \frac{C15}{n^4(1-eps)^4}$

$lnnm2f = -1/2\,nal + 1/2\,nal\,(al\,L - \ln(2)) + \ln(\pi\,nal) + 1/6\,nal^{-1} - 1/45\,n3al^{-1} + \frac{C44}{n4al}$

$EXP = -1/2\,nal\,\ln(2) + n - nL - 1/2\,n2alm1 - 1/6\,n3alm2 - 1/12\,n4alm3 + nal\,L - 1/4\,n5alm4 - 1/2\,\ln(2\,\pi\,n\,(1 - eps)) - 1/12\,\frac{1}{n(1-eps)} + \frac{1}{360}\,\frac{1}{n^3(1-eps)^3} - \frac{C15}{n^4(1-eps)^4} + 1/2\,nal - 1/2\,nal\,(al\,L - \ln(2)) - \ln(\pi\,nal) - 1/6\,nal^{-1} + 1/45\,n3al^{-1} - \frac{C44}{n4al}$

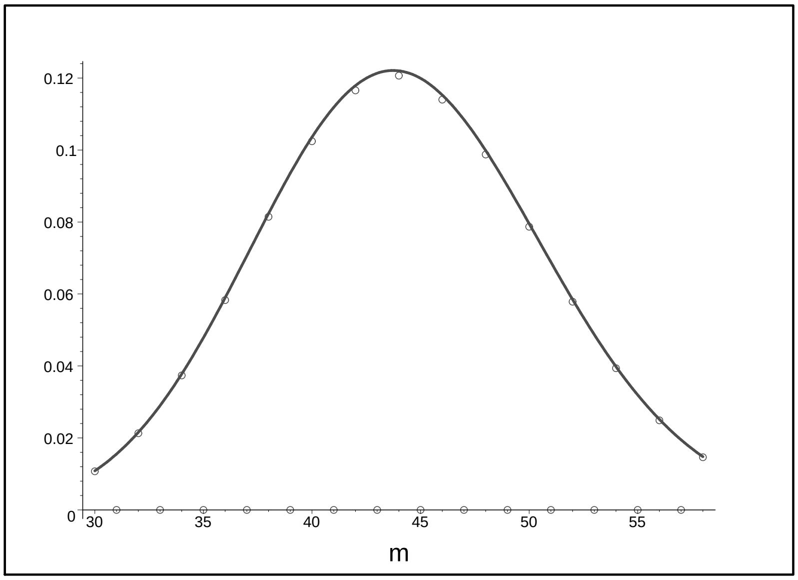

Figure 2: $I(m,n)/I_n$ (circle) and the asymptotic Equ. (10) (line)

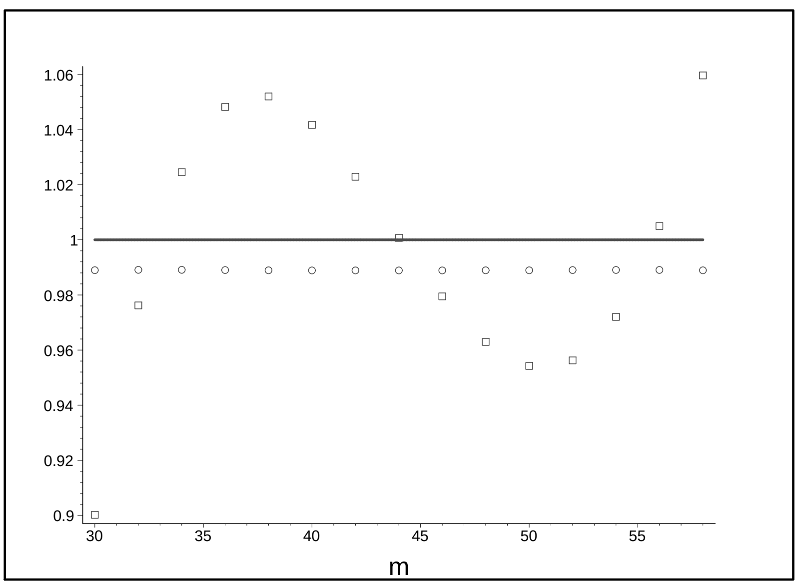

Figure 3: quotient of $I(m,n)/I_n$ and the asymptotic $2\frac{e^{-x^2/2}}{\sqrt{2\pi}\sigma}$ (box) as well as the quotient of $I(m,n)/I_n$ and the asymptotic Equ. (10) (circle)

We use again $f_3(z,m)$. The multiseries's scale is here $n \gg n^\alpha \gg 1/\varepsilon$ if $\alpha > 1/2$ and $n \gg 1/\varepsilon \gg \alpha$ if $\alpha > 1/2$. We set

$$\begin{aligned} \varepsilon &= n^{\alpha-1}, \\ m &= n(1-\varepsilon), \\ L &:= ln(n). \end{aligned}$$

This leads to

$$\begin{aligned} \kappa_1 &= \rho^2 + m, \\ \rho^2 &= n - m = n^\alpha \text{ (we use that relation as a substitution in the sequel) }, \\ \kappa_j &= n^\alpha 2^{j-1}, \\ \rho &= n^{\alpha/2}, \\ \ln(\rho) &= \frac{\alpha}{2} L, \\ \ln(m) &= L - \varepsilon - 1/2\,\varepsilon^2 - 1/3\,\varepsilon^3 - 1/4\,\varepsilon^4 + \mathcal{O}(\varepsilon^5). \end{aligned}$$

The dominant part of (2) gives

$$\begin{aligned} \frac{f_3(\rho)}{\rho^n} &= \exp(E_4), \\ E_4 &= \rho^2/2 + m\ln(\rho) - \ln(m!) - n\ln(\rho) = n^\alpha/2 + (n - n^\alpha)\frac{\alpha}{2}L - \ln(m!) - n\frac{\alpha}{2}L, \\ \ln(m!) &= -n + n^\alpha + n(1-\varepsilon)\ln(m) + \frac{1}{2}\ln(2\pi m) + \frac{1}{12}m^{-1} - \frac{1}{360}m^{-2} + \mathcal{O}\left(\frac{1}{m^3}\right). \end{aligned}$$

Now we use the substitution

$$n\varepsilon^j = n^{j\alpha-(j-1)},$$

this leads to

$$\begin{aligned} n(1-\varepsilon)\ln(m) &= nL - n^\alpha + 1/2\,n^{2\alpha-1} + 1/6\,n^{3\alpha-2} + 1/12\,n^{4\alpha-3} - n^\alpha L + 1/4\,n^{5\alpha-4} + \mathcal{O}(n^{6\alpha-5}), \\ \exp(E_4) &= F_5\exp(E_5), \\ F_5 &= \frac{1}{\sqrt{2\pi n(1-\varepsilon)}}\left[1 - 1/12\,\frac{1}{(1-\varepsilon)\,n} + \frac{1}{160}\,\frac{1}{(1-\varepsilon)^2\,n^2} + \mathcal{O}(\frac{1}{n^3})\right], \\ E_5 &= (1/2 - 1/2\,\alpha\,L + L)\,n^\alpha + (-L+1)\,n \\ &\quad - 1/2\,n^{2\alpha-1} - 1/6\,n^{3\alpha-2} - 1/12\,n^{4\alpha-3} - 1/4\,n^{5\alpha-4} + \mathcal{O}(n^{6\alpha-5}). \end{aligned}$$

Now we turn to the integral. Proceeding as previously, we have

$$\begin{aligned} \frac{1}{2}(n-m)(e^{2\mathbf{i}\theta} - 1 - 2\mathbf{i}\theta) &= -\frac{u^2}{2}, \\ \theta &= \sum_{i=1}^{\infty} a_i \frac{u^i}{n^{\alpha i/2}}, \\ a_1 &= \frac{\sqrt{2}}{2}, \\ a_2 &= -\frac{\mathbf{i}}{6} \end{aligned}$$

This integration gives

$$\frac{1}{2\sqrt{\pi n^\alpha}} F_6,$$
$$F_6 = 1 - 1/6 \frac{1}{n^\alpha} + \frac{1}{72} \frac{1}{n^{2\alpha}} + \mathcal{O}\left(\frac{1}{n^{3\alpha}}\right).$$

Finally

$$\begin{aligned}
\frac{I(m,n)}{n!} &= \exp(E_5) F_7, \\
F_7 &= \frac{1}{2\sqrt{\pi n^\alpha}} F_6 F_5 = \frac{1}{2\sqrt{\pi n^\alpha}} \left(1 - 1/6\, n^{\alpha-1} + \frac{1}{72} \frac{1}{n^{2\alpha}} + \mathcal{O}\left(\frac{1}{n^{3\alpha}}\right)\right) \times \\
&\times \frac{1}{\sqrt{2\pi n(1-\varepsilon)}} \left[1 - 1/12 \frac{1}{(1-\varepsilon)\, n} + \frac{1}{160} \frac{1}{(1-\varepsilon)^2 n^2} + \mathcal{O}(\frac{1}{n^3})\right], \\
E_5 &= (1/2 - 1/2\, \alpha\, L + L)\, n^\alpha + (-L+1)\, n \\
&- 1/2\, n^{2\alpha-1} - 1/6\, n^{3\alpha-2} - 1/12\, n^{4\alpha-3} - 1/4\, n^{5\alpha-4} + \mathcal{O}(n^{6\alpha-5}).
\end{aligned}$$

The choice of θ_0 is here $n^{-5\alpha/12}$, $(\kappa_1, \kappa_2 = \mathcal{O}(n^\alpha))$. The tail pruning is the same as for $I(m,n)$. Finally, we obtain the following asymptotic result:

Theorem 4.1 *The asymptotic expression of the $I(m,n)$ for large deviation $m = n - n^\alpha, 0 < \alpha < 1$ is given by*

$$\begin{aligned}
\frac{I(m,n)}{n!} &= 2 \exp(E_5) F_7, \\
&= 2 \frac{1}{2\sqrt{\pi n^\alpha}} \left(1 - 1/6\, n^{\alpha-1} + \frac{1}{72} \frac{1}{n^{2\alpha}} + \mathcal{O}\left(\frac{1}{n^{3\alpha}}\right)\right) \times \\
&\times \frac{1}{\sqrt{2\pi n(1-\varepsilon)}} \left[1 - 1/12 \frac{1}{(1-\varepsilon)\, n} + \frac{1}{160} \frac{1}{(1-\varepsilon)^2 n^2} + \mathcal{O}(\frac{1}{n^3})\right] \times \\
&\times \exp\left[(1/2 - 1/2\, \alpha\, L + L)\, n^\alpha + (-L+1)\, n \right. \\
&\left. -1/2\, n^{2\alpha-1} - 1/6\, n^{3\alpha-2} - 1/12\, n^{4\alpha-3} - 1/4\ n^{5\alpha-4} + \mathcal{O}(n^{6\alpha-5})\right]. \qquad (11)
\end{aligned}$$

Note that we prefer to keep two separate factors in (11): one in powers of n^α and one in powers of n instead of mixing them.

Let us analyze the importance of the terms in E_5. We have two sets: the set A of dominant terms, which stay in the exponent and the set B of small terms, leading to a coefficient of type $(1 + \Delta)$, with Δ small. The property of each term may depend on α. In E_5, each term $n^{j\alpha-(j-1)}$ is in A if $j > \frac{1}{1-\alpha}$ and in B otherwize. We finally mention that our non-central range is not sacred: other types of ranges can be analyzed with similar methods.

To check the quality of our asymptotics, we have first chosen $n = 2000$ and a range $\alpha \in (0.125, 0.45)$. This corresponds to the range $m \in (1968, 1998)$.

Figure 4 gives $\ln(I(n,m)/n!)$ (circle) and ln(Equ.(11))(line) with the substitutions $n^\alpha = n - m, \varepsilon = (n-m)/n, \alpha = \ln(n-m)/ln(n), n^{2\alpha-1} = (n-m)^2/n, \ldots$.

Figure 5 gives the quotient of $\ln(I(n,m)/n!)$ and ln(Equ.(11)) (circle).

Another way for checking the quality is to fix α. We choose $\alpha = 1/4$ and $n \in (1950, 2000)$. Of course, we must use an integer value for m: $m = \lfloor n - n^\alpha + 1 \rfloor$. Hence, in (11), we use α as the root of $n - n^\alpha - m = 0$.

Figure 6 gives $\ln(I(n,m)/n!)$ (circle) and ln(Equ.(11)) (line). The relative error is of order 5.10^{-4}.

Figure 7 gives the quotient of $\ln(I(n,m)/n!)$ and ln(Equ.(11)) (circle).

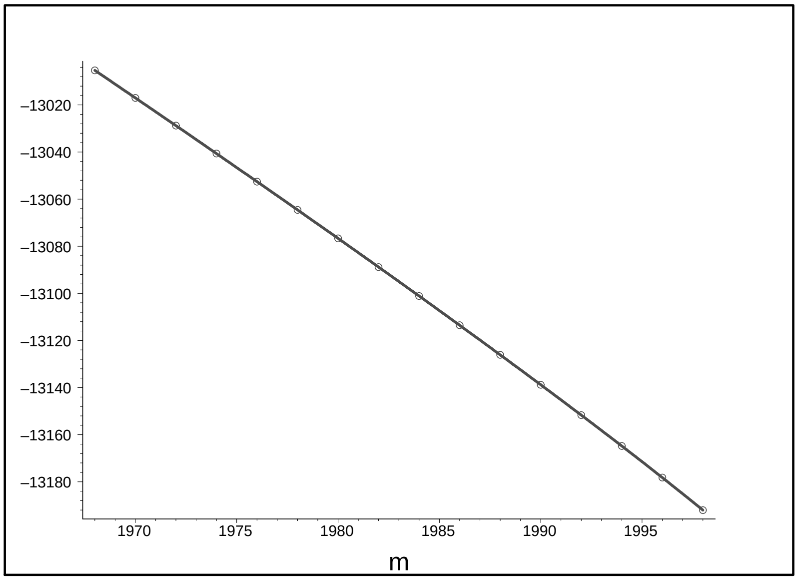

Figure 4: $n = 2000$, $\ln(I(n,m)/n!)$ (circle) and ln(Equ.(11)) (line)

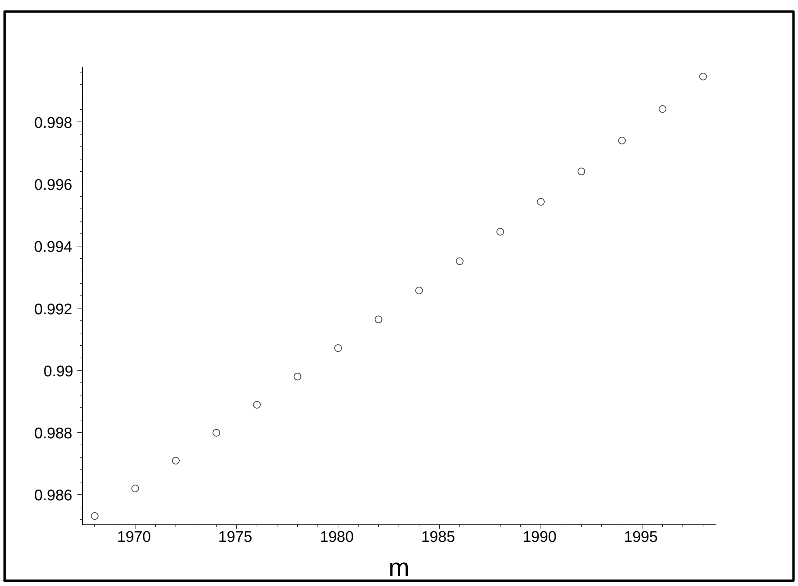

Figure 5: $n = 2000$, quotient of $\ln(I(n,m)/n!)$ and ln(Equ.(11)) (circle)

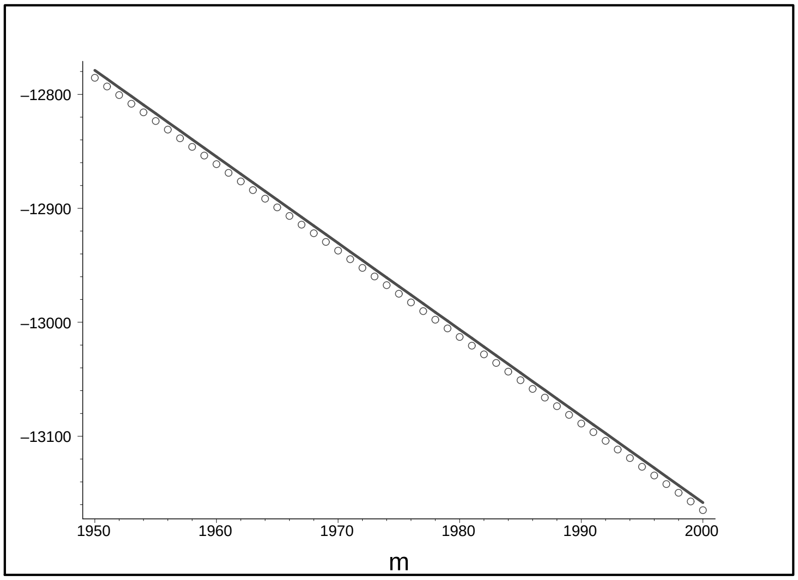

Figure 6: $\alpha = 1/4$, $\ln(I(n,m)/n!)$ (circle) and ln(Equ.(11)) (line)

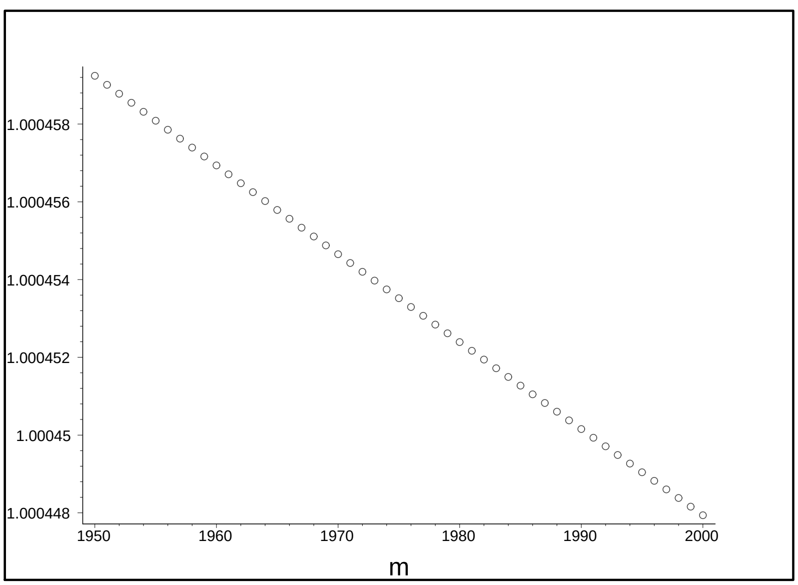

Figure 7: $\alpha = 1/4$, quotient of $\ln(I(n,m)/n!)$ and ln(Equ.(11)) (circle)

5 Appendix. Justification of the integration procedure

The tail pruning for the Gaussian asymptotics leads to analyze

$$\Re[\ln(f_3(\rho e^{\mathbf{i}\theta}) - n\mathbf{i}\theta] = \Re[\rho^2 e^{2\mathbf{i}\theta}/2 + m\mathbf{i}\theta - n\mathbf{i}\theta] = \frac{\rho^2\cos(2\theta)}{2}$$

which has two dominant peaks at 0 and π. So we must be more precise, we have, with $n-m$ even,

$$\Re\left[\mathrm{e}^{1/2\,\rho^2\mathrm{e}^{2\,\mathbf{i}\theta}+\mathbf{i}(m-n)\theta}\right] = \mathrm{e}^{1/2\,\rho^2\cos(2\,\theta)}\cos\left(1/2\,\rho^2\sin\left(2\,\theta\right)+\theta\left(m-n\right)\right),$$

and, indeed, if we set $\theta = \pi + \delta$, we recover the same expression. Hence a factor 2 in (10). The large deviation case leads to the same analysis.

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