SUM OF POSITIONS OF RECORDS IN RANDOM PERMUTATIONS: ASYMPTOTIC ANALYSIS

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ABSTRACT. This statistic, i.e. the sum of positions of records, has been the object of recent interest in the literature. Using the saddle point method, we obtain from the generating function of the sum of positions of records in random permutations and Cauchy's integral formula, asymptotic results in central and non-central regions. In the non-central region, we derive asymptotic expansions generalizing some results by Kortchemski. In the central region, we obtain a limiting distribution related to Dickman's function. This paper fits within the framework of Analytic Combinatorics.

1. INTRODUCTION

The statistic **srec** is defined as the sum of positions of records in random permutations. The generating function (GF) of **srec** is given by

$$G(z) = \prod_{i=1}^{n}(z^i + i - 1), \tag{1}$$

and the probability generating function (PGF) is given by

$$Z(z) = \frac{\prod_{i=1}^{n}(z^i + i - 1)}{n!}. \tag{2}$$

This statistic has been the object of recent interest in the literature. Let us mention Kortchemski [9], where he obtains the GF (1) and also proves that, in a large deviation domain,

$$J_n(\ell) := [z^\ell]G(z) \sim e^{n\ln(n)y + \mathcal{O}(n)}, \text{ where } \ell = \frac{n(n+1)}{2}x, \quad x = 1 - y^2, \quad 0 < y < 1, \tag{3}$$

with an error $\mathcal{O}(1/\ln(n))$ (we assume that ℓ is integer). [1] The GF (1) is proved as follows: let X_k be 1 if k is a position of a record and 0 otherwise. It is well known (see, for instance, Rényi [16]), that $X_1, X_2, .., X_n$ are independent and $\mathbb{P}(X_k = 1) = \frac{1}{k}$, hence (1).

In [15], Prodinger analyzes the same statistic for a series of n random geometric variables (with weak and strong records). He obtains an asymptotic expansion of $\mathbb{E}(\textbf{srec})$.

In [8], Knopfmacher and Mansour analyze **srec** in a random composition of an integer m. Let us also mention Kuba and Panholzer [10] which includes a related parameter.

In this paper, we obtain, in Section 2 an asymptotic expansion generalizing the large deviation result (3) and we analyze the central region $\ell = yn$ in Section 3. We use Cauchy's integral formula, the saddle point method and a technique developped in Arratia, Barbour and Tavare [2]. Section 4 provides the justification of the integration procedures used in the saddle point method. Section 5 concludes the paper.

Our results can be summarized as follows

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[1] As advocated by D.E. Knuth, we denote by $[z^n]f(z)$ the coefficient of z^n in the power expansion of $f(z)$.

Theorem 1.1. *Let the large deviation domain be defined as* $\ell = \frac{n(n+1)}{2}x, \quad x = 1-y^2, \quad 0 < y < 1$. *The asymptotic value of* $J_n(\ell) := [z^\ell]G(z)$*, in this large deviation domain, is given by*

$$J_n(\ell) \sim \tilde{J}_n(\ell) = \frac{e^{S(\tilde{z})}}{\sqrt{2\pi S''(\tilde{z})}}$$

where, with $L := \ln(n)$, $S(\tilde{z}), S''(\tilde{z})$ *are given by*

$$S(\tilde{z}) = \Sigma_3 + \Sigma_4 = nyL + n\left[y(-1+\ln(y)) + \frac{6191y}{3600L} + \dots\right] + \frac{n^2}{n^{1/y}L}\left[-\frac{y}{y^{1/y}} + \dots\right] + \dots,$$

$$S''(\tilde{z}) = n^3\left[\frac{4y^3}{5L} - \frac{4y^3(-18+5\ln(y))}{25L^2} + \dots\right].$$

Theorem 1.2. *In the central region,* $V := \mathbf{srec}/n$ *converges in distribution to a random variable with density given by*

$$e^{-\gamma}\rho(v),$$

where $\rho(v)$ *is the Dickman's function.*

An alternative (non-constructive) proof of Theorem 1.2 goes as follows. Recall that $\mathbf{srec} = \sum_{k=1}^n k\, X_k$ with X_k the indicator that k is a position of a record. Rényi [16] proved that the X_k are independent with $\mathbb{P}(X_k = 1) = 1/k$. Hence **srec** is the sum of independent random variables with known distribution and, as noted by Hwang and Tsai [7], the argument provided in Corollary 2.8 of Arratia et al. [1] actually yields the local limit theorem

$$\mathbb{P}(\mathbf{srec} = k) \sim \frac{e^{-\gamma}\rho(k/n)}{n} \tag{4}$$

The connection between Corollary 2.8 of Arratia et al. [1] and (4) is not transparent. However, in a curious coincidence, a detailed explicit proof of (4) – relying on techniques which are independent of ours – has recently been published to the ArXiv, see Giuliano, Szewczak and Weber [5][2].

This paper fits within the framework of Analytic Combinatorics. A preliminary version was presented at the AofA 2013 Conference.

2. The large deviation $\ell = \frac{n(n+1)}{2}(1-y^2)$

2.1. **The Saddle point method.**

By Cauchy's theorem,

$$\begin{aligned} J_n(\ell) &= \frac{1}{2\pi\mathbf{i}}\int_\Omega \frac{G(z)}{z^{\ell+1}}dz \\ &= \frac{1}{2\pi\mathbf{i}}\int_\Omega e^{S(z)}dz, \end{aligned} \tag{5}$$

where Ω is inside the analyticity domain of the integrand and encircles the origin and

$$S(z) = \sum_{i=1}^n \ln(z^i + i - 1) - \left(\frac{n(n+1)}{2}(1-y^2)+1\right)\ln(z).$$

Set

$$S^{(i)} := \frac{d^i S}{dz^i}.$$

We will use the Saddle point method (for a good introduction to this method, see Flajolet and Sedgewick [4, ch.$VIII$]). First we must find the solution of

$$S^{(1)}(\tilde{z}) = 0$$

[2] We thank Y. Swan for providing these lines of proof.

with smallest modulus. This leads to

$$\sum_{i=1}^{n} \frac{i\tilde{z}^i}{\tilde{z}^i + i - 1} - \left(\frac{n(n+1)}{2}(1-y^2) + 1 \right) = 0. \tag{6}$$

The left-hand side of this equation is an increasing function of $\tilde{z}$, so the solution $\tilde{z}$ is unique. It is easy to check that, for large n (depending on y), $0 < \tilde{z} < 2$. For instance, for $y = 0.1$, this is true for $n \geq 200$, for $y = 0.2$, this is true for $n \geq 60$.

In some previous papers (see Louchard and Prodinger [13], [11], [14]), we simply tried $\tilde{z} = z^* + \varepsilon$ for some z^*, plugged into (6), and expanded into ε. Here it appears that we cannot get this expansion. So we expand first (6) itself. But we must be careful. There exists some $\tilde{i}$ such that $\tilde{z}^{\tilde{i}} = \tilde{i}$. Some numerical experiments suggest that $\tilde{i} = \mathcal{O}(n)$. So we set $\tilde{i} = \alpha n, 0 < \alpha < 1$ and we must now compute α. We obtain $\tilde{z} = e^{\xi} > 1$, with

$$0 < \xi = \frac{L + \ln(\alpha)}{\alpha n} = o(1),$$

where here and in the sequel, $L := \ln(n)$. Note that this leads to

$$\tilde{z}^n = \exp\left(\frac{L + \ln(\alpha)}{\alpha} \right) = n^{1/\alpha} \alpha^{1/\alpha}.$$

We use the classical splitting of the sum technique, (see for instance Greene and Knuth, [6]). Let us assume $\tilde{i}$ integer, or use instead $\lfloor \tilde{i} \rfloor$ (we keep the notation $\tilde{i}$ in the sequel, to simplify our expressions). Now equ. (6), leads to (we provide in the sequel only a few terms in the expansions, but Maple knows and uses more)

$$\Sigma_1 := \sum_{i=1}^{\tilde{i}} \frac{i\tilde{z}^i}{\tilde{z}^i + i - 1},$$

$$\Sigma_2 := \sum_{i=\tilde{i}+1}^{n} \frac{i\tilde{z}^i}{\tilde{z}^i + i - 1} - \left(\frac{n(n+1)}{2}(1-y^2) + 1 \right).$$

As

$$\frac{\tilde{z}^i - 1}{i} < \frac{\tilde{z}^{\tilde{i}} - 1}{\tilde{i}} < \frac{\tilde{z}^{\tilde{i}}}{\tilde{i}} = 1, i < \tilde{i},$$

we have

$$\Sigma_1 = \sum_{i=1}^{\tilde{i}} \frac{\tilde{z}^i}{1 + \frac{\tilde{z}^i - 1}{i}} = \sum_{i=1}^{\tilde{i}} \tilde{z}^i \left[1 - \frac{\tilde{z}^i - 1}{i} + \left(\frac{\tilde{z}^i - 1}{i} \right)^2 + \dots \right] \tag{7}$$

The first summation is immediate

$$\sum_{i=1}^{\tilde{i}} \tilde{z}^i = \frac{\tilde{z}^{\tilde{i}+1} - 1}{\tilde{z} - 1} - \frac{\tilde{z}}{\tilde{z} - 1} \sim \frac{\alpha^2 n^2}{L + \ln(\alpha)} + \mathcal{O}(n). \tag{8}$$

For the next summations, we use the Euler-Maclaurin summation formula. First of all, the correction (to first order) arising from replacing the summations in (7) by integrals is given by

$$\frac{1}{2} + \frac{1}{2} \frac{\tilde{i}}{1 + \frac{\tilde{i} - 1}{\tilde{i}}} = \frac{1}{2} + \frac{1}{2} \frac{\alpha n}{2 - 1/(\alpha n)} \sim \frac{1}{4} \alpha n. \tag{9}$$

Next, we must compute integrals such as

$$\int_1^{\tilde{i}} \tilde{z}^v \left(\frac{\tilde{z}^v - 1}{v} \right)^k dv. \tag{10}$$

But we know that

$$\begin{aligned}
K &:= \int_1^{\tilde{i}} \tilde{z}^v \left(\frac{\tilde{z}^v - 1}{v}\right) dv = \int_1^{\tilde{i}} e^{\xi v} \left(\frac{e^{\xi v} - 1}{v}\right) dv = \int_1^{\tilde{i}} \left[e^{2\xi v} - e^{\xi v}\right] \frac{dv}{v} \\
&= \int_{-1}^{-\tilde{i}} \left[e^{-2\xi u} - e^{-\xi u}\right] \frac{du}{u} = \int_{-\xi}^{-\xi\tilde{i}} \left[e^{-2t} - e^{-t}\right] \frac{dt}{t} \\
&= \mathbf{Ei}(1, -2\xi) - \mathbf{Ei}(1, -\xi) + \mathbf{Ei}(1, -\tilde{i}\xi) - \mathbf{Ei}(1, -2\tilde{i}\xi) \\
&= \mathbf{Ei}(1, -2\xi) - \mathbf{Ei}(1, -\xi) + \mathbf{Ei}(1, -(L + \ln(\alpha))) - \mathbf{Ei}(1, -2(L + \ln(\alpha))),
\end{aligned}$$

where $\mathbf{Ei}(1, x)$ is the exponential integral of index 1 and we use suitable extensions (with Cauchy principal values):

$$\mathbf{Ei}(1, x) := \int_x^{\infty} \frac{e^{-y}}{y} dy.$$

Setting $L_1 := L + \ln(\alpha) \succ 1$, [3] we have

$$\begin{aligned}
\Re(\mathbf{Ei}(1, -\xi)) &= -\gamma - \ln(\xi) - \xi - \frac{\xi^2}{4} + \dots, \\
\Re(\mathbf{Ei}(1, -L_1)) &= e^{L_1} \left[-\frac{1}{L_1} - \frac{1}{L_1^2} + \dots\right].
\end{aligned}$$

This gives, for instance,

$$K \sim -\ln(2\xi) - \xi - \frac{3\xi^2}{4} + \dots + \ln(\xi) - e^{2L_1} \left[-\frac{1}{2L_1} - \frac{1}{4L_1^2} + \dots\right] + e^{L_1} \left[-\frac{1}{L_1} - \frac{1}{L_1^2} + \dots\right].$$

But $e^{L_1} = n\alpha$. Proceeding to asymptotics w.r.t n and to asymptotics w.r.t L, we derive

$$K \sim n^2 \left[\frac{\alpha^2}{2L} + \frac{1}{8} \frac{\alpha^2(-4\ln(\alpha) + 2)}{L^2} + \mathcal{O}\left(\frac{1}{L^3}\right)\right] + \mathcal{O}(n).$$

We use similar expansions for terms like (10), using integration by parts. This finally leads, with (9) and (8), to

$$\Sigma_1 = n^2 \left[\frac{47\alpha^2}{60L} + \frac{\alpha^2(-564\ln(\alpha) - 155)}{720L^2} + \mathcal{O}(L^{-3})\right] + \mathcal{O}(n). \tag{11}$$

Note that we must use enough terms in (7) to obtain a sufficient precision: 5 terms are necessary to obtain the first two terms of (11).

Now we turn to Σ_2. As

$$\frac{i-1}{\tilde{z}^i} < \frac{\tilde{i}+1-1}{\tilde{z}^{\tilde{i}+1}} < \frac{\tilde{i}}{\tilde{z}^i} = 1, i > \tilde{i},$$

we have

$$\begin{aligned}
\Sigma_2 &= \sum_{i=\tilde{i}+1}^{n} \frac{i}{1 + \frac{i-1}{\tilde{z}^i}} - \left(\frac{n(n+1)}{2}(1 - y^2) + 1\right) \\
&= \sum_{i=\tilde{i}+1}^{n} i \left[1 - \frac{i-1}{\tilde{z}^i} + \left(\frac{i-1}{\tilde{z}^i}\right)^2 + \dots\right] - \left(\frac{n(n+1)}{2}(1 - y^2) + 1\right).
\end{aligned}$$

After all summations and substitutions such as

$$\tilde{z}^{in} = (n^{1/\alpha}\alpha^{1/\alpha})^i, \tilde{z}^{in\alpha} = (n\alpha)^i,$$

[3] We recall the notation: $f(n) \prec g(n) \Longleftrightarrow \lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$.

we obtain

$$\Sigma_2 = n^2\left[\frac{1}{2} - \frac{\alpha^2}{2} - \frac{47\alpha^2}{60L} + \frac{\alpha^2(2820\ln(\alpha) - 5839)}{3600L^2} + \dots\right] + \mathcal{O}(n)$$
$$+ \frac{n^3}{n^{1/\alpha}}\left[\frac{\alpha}{\alpha^{1/\alpha}L} + \dots\right] - \left(\frac{n(n+1)}{2}(1-y^2) + 1\right) + \dots.$$

Note the presence of the same coefficient $\frac{47}{60}$ in Σ_1 and Σ_2.

So

$$S'(\tilde{z}) = \Sigma_1 + \Sigma_2 = n^2\left[\frac{1}{2} - \frac{\alpha^2}{2} - \frac{3307\alpha^2}{1800L^2} + \dots - (1-y^2)/2\right] + \mathcal{O}(n) + \frac{n^3}{n^{1/\alpha}}\left[\frac{\alpha}{\alpha^{1/\alpha}L} + \dots\right] + \dots = 0, \tag{12}$$

Putting the coefficient of n^2 to 0, and solving wrt α gives the solution

$$\alpha^* = y - \frac{3307y}{1800L^2} + \frac{10936249y}{2160000L^4} + \dots \tag{13}$$

Now we must consider the other terms of (12). First we must compare n with $\frac{n^3}{n^{1/\alpha}}$. If $\alpha > \frac{1}{2}$, $n^{3-1/\alpha} > n$ and vice-versa. The most interesting case is the case $\alpha > \frac{1}{2}$ï£¡ (the other one can be treated by similar methods). Note that there are also other terms in (12) of order $n^{k-(k-2)/\alpha}, k \geq 4$. These terms are greater than n if $\alpha > (k-2)/(k-1)$.

Returning to (12), we first compute $n^{1/\alpha^*} = n^{1/y}\varphi(y,L)$, with

$$\varphi(y,L) = e^{L(1/\alpha^* - 1/y)} = 1 + \frac{3307}{1800yL} + \dots \tag{14}$$

So we obtain from (12) the term

$$\frac{n^3}{n^{1/y}\varphi(y,L)}\left[\frac{\alpha^*}{\alpha^{*1/\alpha^*}L} + \dots\right],$$

and with (13),

$$\frac{n^3}{n^{1/y}}\left[\frac{y}{y^{1/y}L} + \dots\right].$$

Now we set $\alpha = \alpha^* + \frac{Cn}{n^{1/y}}$, plug into (12) (ignoring the $\mathcal{O}(n)$ term), and expand. The n^2 term must theoretically be 0. Actually, it is given by a series of large powers of $1/L$ as we only use a finite number of terms in (13). Solving the coefficient of $\frac{n^3}{n^{1/y}}$ wrt C, we obtain

$$C^* = \frac{1}{Ly^{1/y}} + \dots$$

and

$$\alpha = \alpha^* + \frac{C^*n}{n^{1/y}} + \dots \tag{15}$$

This implies, for instance,

$$\ln(\tilde{z}) = \frac{L + \ln(\alpha)}{\alpha n} \sim \frac{L + \ln(\alpha^* + C^*n^{1-1/y})}{\alpha n} \sim \frac{L + \ln(\alpha^*) + C^*n^{1-1/y}/\alpha^*}{(\alpha^* + C^*n^{1-1/y})n}$$
$$\sim \frac{1}{n}\left[\frac{L + \ln(\alpha^*)}{\alpha^*} - \frac{C^*n^{1-1/y}(-1 + L + \ln(\alpha^*))}{\alpha^{*2}} + \dots\right].$$

2.2. **The computation of $S(\tilde z)$.** Proceeding as above, we have

$$S(\tilde z)=\Sigma_3+\Sigma_4$$

with

$$\Sigma_3=\sum_{i=1}^{\tilde i}\ln(\tilde z^i+i-1) \tag{16}$$

$$=\sum_{i=1}^{\tilde i}\ln(i)+\sum_{i=1}^{\tilde i}\ln\left(1+\frac{\tilde z^i-1}{i}\right)$$

$$=\sum_{i=1}^{\tilde i}\ln(i)+\sum_{i=1}^{\tilde i}\left[\frac{\tilde z^i-1}{i}-\frac{1}{2}\left(\frac{\tilde z^i-1}{i}\right)^2+\dots\right] \tag{17}$$

and substituting $\tilde z=e^{\xi}$,

$$\Sigma_3=n\left[\alpha L+\alpha(\ln(\alpha)-1)+\frac{31\alpha}{36L}+\dots\right].$$

Next,

$$\begin{aligned}\Sigma_4&=\sum_{i=\tilde i+1}^{n}\ln(\tilde z^i+i-1)-\left(\frac{n(n+1)}{2}(1-y^2)+1\right)\ln(\tilde z)\\&=\sum_{i=\tilde i+1}^{n}\ln(\tilde z^i)+\sum_{i=\tilde i+1}^{n}\ln\left(1+\frac{i-1}{\tilde z^i}\right)-\left(\frac{n(n+1)}{2}(1-y^2)+1\right)\ln(\tilde z)\\&=\sum_{i=\tilde i+1}^{n}i\xi+\sum_{i=\tilde i+1}^{n}\left[\frac{i-1}{\tilde z^i}-\frac{1}{2}\left(\frac{i-1}{\tilde z^i}\right)^2+\dots\right]-\left(\frac{n(n+1)}{2}(1-y^2)+1\right)\xi\\&=nL\left[\frac{-\alpha^2+y^2}{2\alpha}-\frac{1}{2}\frac{(\alpha^2-y^2)\ln(\alpha)}{L\alpha}+\dots\right]+\frac{n^2}{n^{1/\alpha}}\left[-\frac{(\alpha+L+\ln(\alpha))\alpha}{(L+\ln(\alpha))^2\alpha^{1/\alpha}}+\dots\right]+\dots,\end{aligned}$$

and, using (15), (14),

$$S(\tilde z)=\Sigma_3+\Sigma_4=nyL+n\left[y(-1+\ln(y))+\frac{6191y}{3600L}+\dots\right]+\frac{n^2}{n^{1/y}L}\left[-\frac{y}{y^{1/y}}+\dots\right]+\dots \tag{18}$$

2.3. **The computation of $S''(\tilde z)$.** We have

$$S''(\tilde z)=\sum_{i=1}^{n}\left[\frac{\tilde z^i i^2}{\tilde z^2(\tilde z^i+i-1)}-\frac{\tilde z^i i}{\tilde z^2(\tilde z^i+i-1)}-\frac{\tilde z^{2i}i^2}{\tilde z^2(\tilde z^i+i-1)^2}\right]+\left(\frac{n(n+1)}{2}(1-y^2)+1\right)/\tilde z^2.$$

Proceeding as above and omitting the details, we have (here we only use the n^3 term)

$$S''(\tilde z)=n^3\left[\frac{4y^3}{5L}-\frac{4y^3(-18+5\ln(y))}{25L^2}+\dots\right]. \tag{19}$$

Similarly, we have

$$S'''(\tilde z)=\mathcal{O}\left(\frac{n^4}{L}\right),$$

$$S^{(4)}(\tilde z)=\mathcal{O}\left(\frac{n^5}{L}\right).$$

2.4. **The final integration.** Now we obtain

$$J_n(\ell)=\frac{1}{2\pi\mathbf{i}}\int_{\Omega'}\exp\left[S(\tilde{z})+S^{(2)}(\tilde{z})(z-\tilde{z})^2/2!+\sum_{k=3}^{\infty}S^{(k)}(\tilde{z})(z-\tilde{z})^k/k!\right]dz+\frac{1}{2\pi\mathbf{i}}\int_{\Omega-\Omega'}\exp[S(z)]dz$$

where Ω' is in the domain of convergence of the $S(z)$ function power expansion (note carefully that the linear term vanishes).

In Figure 1 we show, for $n=150, y=0.5, z=x+\mathbf{i}\tau$,

$$H(z)=\Re(S(z))=\Re\left(\sum_{i=1}^{n}\ln(z^i+i-1)-\left(\frac{n(n+1)}{2}(1-y^2)+1\right)\ln(z)\right).$$

In Figures 2 and 3 we show, for $n=1000, y=0.5$,

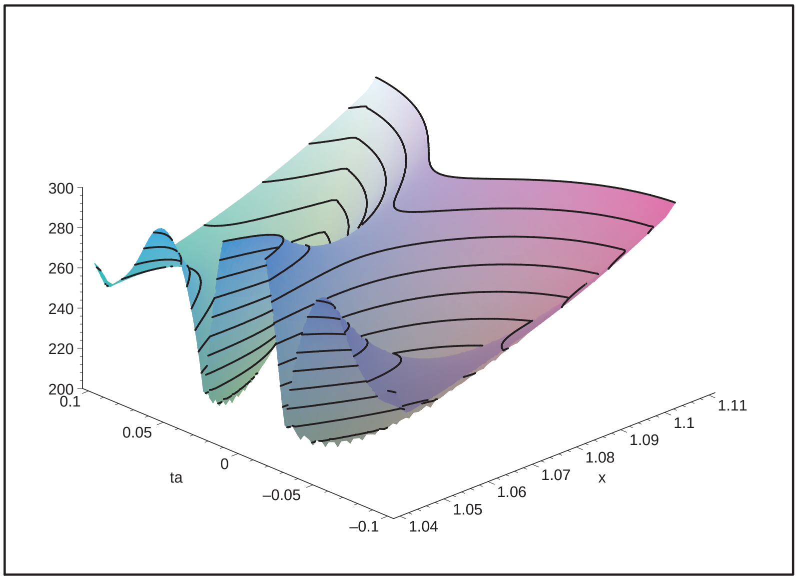

FIGURE 1. $H(z)$, $n=150, y=0.5$

$$\Re\left[H_1\left(e^{\xi+\mathbf{i}\theta}\right)-H_1\left(e^{\xi}\right)\right]$$

as function of θ, with

$$H_1(z)=\sum_{i=1}^{n}\ln(z^i+i-1).$$

From these Figures and some numerical results, the decreasing property along $e^{\xi+\mathbf{i}\theta}$ is clear. It appears that this function is firstly rapidly decreasing and, next, asymptotically given by a large (negative) constant. We will prove these facts in Sec. 4.

As usual, we want to introduce a splitting value θ_0 such that $n^3\theta_0^2\to\infty, n^4\theta_0^3\to 0, n\to\infty$. For instance, we choose $\theta_0=n^{\beta}, \beta=-\frac{17}{12}$.

Proceeding to tails pruning, we want to find asymptotics for sums such that (see for instance, (17)),

$$\Re\left[\sum_{1}^{\alpha n}\ln\left[\frac{\exp(k[\xi+\mathbf{i}\theta])-1}{k}\right]\right]$$

of a suitable form

$$\frac{n^3}{L}\varphi_1(\theta).$$

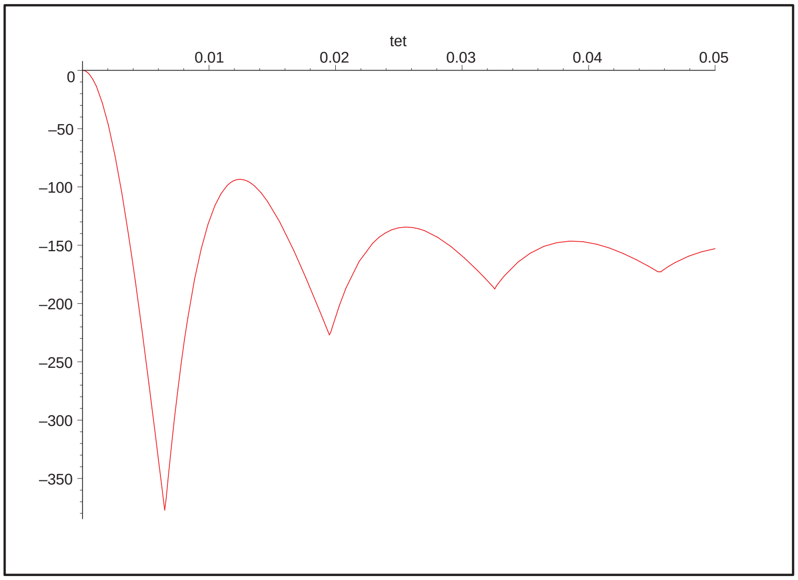

FIGURE 2. $\Re\left[H_1\left(e^{\xi+\mathbf{i}\theta}\right)-H_1\left(e^{\xi}\right)\right], n=1000, y=0.5, \theta=0..0.05$

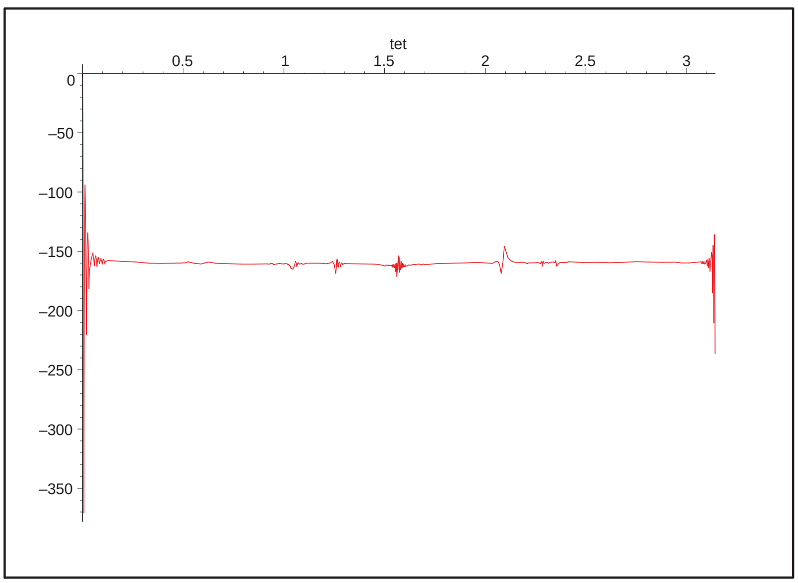

FIGURE 3. $\Re\left[H_1\left(e^{\xi+\mathbf{i}\theta}\right)-H_1\left(e^{\xi}\right)\right], n=1000, y=0.5, \theta=0..\pi$

This should entail that

$$\int_{\theta_0}^{2\pi-\theta_0} e^{\frac{n^3}{L}\varphi_1(\theta)} d\theta$$

is negligible. This is necessary to be sure that we can ignore, in (5), the integration outside the domain $[-\theta_0, \theta_0]$. See again Flajolet and Sedgewick [4, ch.$VIII$]. The details are given in Sec.4.

So we choose $\Omega' = \tilde{z}\exp \mathbf{i}\theta, \theta \in [-\theta_0, \theta_0]$. We derive

$$J_n(\ell) \sim \frac{1}{2\pi}\int_{-\theta_0}^{\theta_0} \exp\left[S(\tilde{z}) + S^{(2)}(\tilde{z})(z-\tilde{z})^2/2! + S^{(3)}(\tilde{z})(z-\tilde{z})^3/3! + \mathcal{O}(n^5\theta_0^4)\right]\tilde{z}d\theta$$

$$\sim \frac{1}{2\pi}\int_{-\theta_0}^{\theta_0} \exp\left[S(\tilde{z}) + S^{(2)}(\tilde{z})(z-\tilde{z})^2/2! + S^{(3)}(\tilde{z})(z-\tilde{z})^3/3! + \mathcal{O}(n^{-2/3})\right]\tilde{z}d\theta.$$

We can shift the path to a path parallel to the imaginary axis. Set $z = \tilde{z} + \mathbf{i}\tau$. Classically completing the tails, this gives

$$J_n(\ell) \sim \frac{1}{2\pi}\exp[S(\tilde{z})]\int_{-\infty}^{\infty} \exp\left[S^{(2)}(\tilde{z})(\mathbf{i}\tau)^2/2! + S^{(3)}(\tilde{z})(\mathbf{i}\tau)^3/3!\right]d\tau. \tag{20}$$

We can now compute (20), for instance by using the classical trick of setting

$$S^{(2)}(\tilde{z})(\mathbf{i}\tau)^2/2! + \sum_{k=3}^{4} S^{(k)}(\tilde{z})(\mathbf{i}\tau)^k/k! = -u^2/2,$$

computing τ as a truncated series in u which gives (by inversion)

$$\tau = \frac{\sqrt{L}}{n^{3/2}}\left[u\tau_1 + u^2\tau_2 + u^3\tau_3 + u^4\tau_4 + \ldots\right],$$

with

$$\tau_1 = \sqrt{\frac{5}{9y^2}} + \ldots$$

$$\tau_2 = \mathcal{O}\left(\frac{\mathbf{i}\sqrt{L}}{\sqrt{n}}\right),$$

$$\tau_3 = \mathcal{O}\left(\frac{L}{n}\right),$$

$$\tau_4 = \mathcal{O}\left(\frac{\mathbf{i}L^{3/2}}{n^{3/2}}\right).$$

Setting $d\tau = \frac{d\tau}{du}du$, expanding w.r.t. n and integrating on $u \in (-\infty, \infty)$, we finally obtain

Theorem 2.1. *The asymptotic value of $J_n(\ell) := [z^\ell]G(z)$, in the large deviation domain considered here, is given by*

$$J_n(\ell) \sim \tilde{J}_n(\ell) = \frac{e^{S(\tilde{z})}}{\sqrt{2\pi S''(\tilde{z})}}$$

where $S(\tilde{z}), S''(\tilde{z})$ are given by (18) and (19).

In Figure 4, we give, for $n = 150$, a comparison between $\ln(J_n(\ell))$ (circle) and $\ln(\tilde{J}_n(\ell))$ (line), as a function of y. The fit is quite good, but when y is close to 1. But ℓ is then small and our asymptotics are no more very efficient (we are outside the large deviation range). We also show the first approximation (3): nLy (blue line) which is only efficient for very large n (i.e. when L is large, see the linear n term in (18). Let us note that other large deviation regions can be analyzed by the same method. See for instance Louchard and Prodinger [12].

3. The central region $\ell = yn$

From (2), we see that, as expected, $Z(1) = 1$. Moreover

$$Z'(z) = \sum_{i=1}^{n} \frac{iz^{i-1}}{z^i + i - 1}Z(z),$$

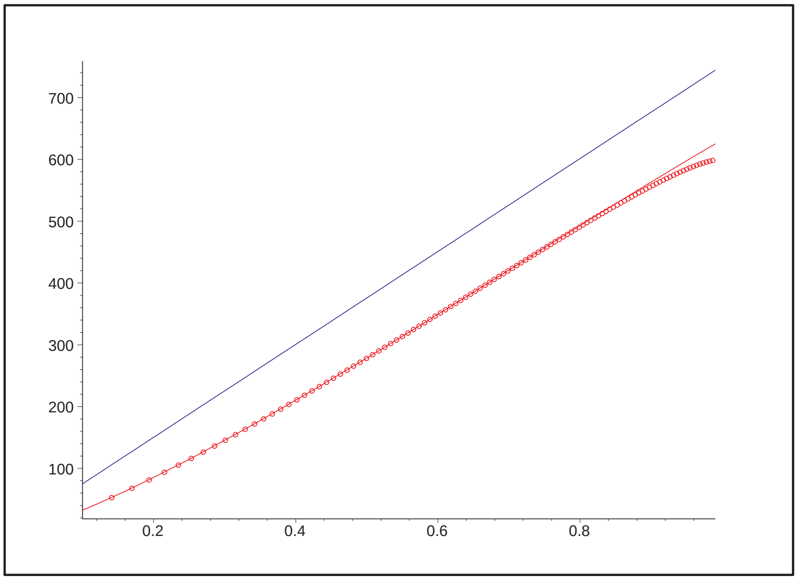

FIGURE 4. Comparison between $\ln(J_n(\ell))$ (circle) and $\ln(\tilde{J}_n(\ell))$ (line), as a function of y, $n = 150$. Also it shows the first approximation (3): nLy (blue line)

and

$$Z''(z) = \sum_{i=1}^{n} \frac{i[z^{i-2}i^2 - 2iz^{i-2} - z^{2i-2} + z^{i-2}]}{(z^i + i - 1)^2} Z(z) + \left(\sum_{i=1}^{n} \frac{iz^{i-1}}{z^i + i - 1}\right)^2 Z(z).$$

So

$$\mathbb{E}(\mathbf{srec}) = Z'(1) = n,$$

and the variance is given by

$$\mathbb{V}(\mathbf{srec}) = Z''(1) + n - n^2 = \sum_{i=1}^{n}(i-2) + n^2 + n - n^2 = \frac{n(n-1)}{2}.$$

Of course $Z(z)$ corresponds to a sum of independent *non*-identically distributed random variables, but it is clear that the Lindeberg-Lévy conditions (see for instance, Feller [3]) are not satisfied here. The distribution is *not* asymptotically Gaussian. We tried to use the classical Saddle point method, but it appears that, at the Saddle point, the second derivative is of order n^2 but the third derivative is of order n^3! So we couldn't apply this method. Fortunately, we can use a technique developped in Arratia, Barbour and Tavare [2], sec. 4.2, which leads to an asymptotic distribution depending on Dickman's function $\rho(x)$[4].

Equ. (2) can be written as

$$Z(z) = \prod_{i=1}^{n}(1 - \frac{1}{i} + \frac{z^i}{i}),$$

which corresponds to a sum of Bernoulli independent random variables, with parameter $1/i$. Let $V := \mathbf{srec}/n$. Then

$$\mathbb{E}\left(e^{-sV}\right) = \exp\left(\sum_{1}^{n} \ln\left[1 + \frac{e^{-is/n} - 1}{i}\right]\right).$$

[4]We are indebted to S.Janson for suggesting this use of Dickman's function.

But $\frac{e^{-is/n}-1}{i} = \mathcal{O}(\frac{1}{n})$ uniformly in i, so

$$\mathbb{E}\left(e^{-sV}\right) = \exp\left(-\sum_1^n \frac{1-e^{-is/n}}{i} + \mathcal{O}(\frac{1}{n})\right). \tag{21}$$

This is exactly the expression given in Arratia et al. [2] p.81, for $\theta = 1$. This leads to an asymptotic distribution given by $e^{-\gamma}\rho(v)$, where $\rho(v)$ is the Dickman's function.

For the sake of completeness, following the lines of [2], let us derive this formula.

By dominated convergence, (21) leads to

$$\begin{aligned}
\mathbb{E}\left(e^{-sV}\right) &\sim \exp\left[-\int_0^1 (1-e^{-sx})\frac{1}{x}dx\right] \\
&= e^{-\gamma}s^{-1}e^{-\mathbf{Ei}(1,s)} \\
&= e^{-\gamma}s^{-1}\sum_0^\infty \frac{(-1)^k}{k!}\left(\int_s^\infty \frac{e^{-y}}{y}dy\right)^k, \qquad (22)
\end{aligned}$$

and, for $k \geq 1$,

$$\begin{aligned}
s^{-1}\left(\int_s^\infty \frac{e^{-y}}{y}dy\right)^k &= \int_1^\infty \ldots \int_1^\infty s^{-1}\exp\left[-s\sum_1^k v_i\right]\frac{dv_1\ldots dv_k}{v_1\ldots v_k} \\
&= \int_1^\infty \ldots \int_1^\infty \int_{\sum_1^k v_i}^\infty e^{-sx}dx\frac{dv_1\ldots dv_k}{v_1\ldots v_k} \\
&= \int_0^\infty e^{-sx}\int_1^\infty \ldots \int_1^\infty [[\sum_1^k v_i < x]]\frac{dv_1\ldots dv_k}{v_1\ldots v_k}dx \\
&= \int_0^\infty e^{-sx}\int \ldots \int_{I_k(x)} \frac{dy_1\ldots dy_k}{y_1\ldots y_k}dx
\end{aligned}$$

where [5]

$$I_k(x) := [[y_1 > x^{-1}, \ldots, y_k > x^{-1}, \sum_1^k y_i < 1]].$$

But the Dickman's function is given by

$$\rho(u) = 1 + \sum_1^\infty \frac{(-1)^k}{k!}\int \ldots \int_{I_k(u)} \frac{dy_1\ldots dy_k}{y_1\ldots y_k},$$

so, by (22) and Vervaat [17], p.90, we obtain the following theorem

Theorem 3.1. *In the central region, $V :=$ **srec**$/n$ converges in distribution to a random variable with density given by*

$$e^{-\gamma}\rho(v).$$

Let us recall that $\rho(u)$ is the solution of

$$u\rho'(u) + \rho(u-1) = 0, \text{ for } u > 0,$$

and

$$\rho(u) = 0, \text{ for } u < 0, \rho(u) = 1, \text{ for } 0 \leq u \leq 1.$$

The first values of $\rho(u)$ are given by

$$\rho(u) = 1 - \ln(u), \text{ for } 1 \leq u \leq 2, \rho(u) = 1 - [1 - \ln(1-u)]\ln(u) + \mathbf{Li}_2(1-u) + \frac{\pi^2}{12} \text{ for } 2 \leq u \leq 3,$$

where

$$\mathbf{Li}_2(z) := -\int_0^z \frac{\ln(1-t)}{t}dt.$$

[5] As advocated by D.E. Knuth, we denote by $[[.]]$ the indicator function: $[[B]] = 1$ if B is true, 0 otherwise.

Note that the function **dilog**(z) often used in Computer algebra systems is given by

$$\mathbf{dilog}(z) = \mathbf{Li}_2(1-z).$$

From (4), we derive

$$\mathbb{P}(\mathbf{srec} = k) \sim \frac{e^{-\gamma}}{n}, k \leq n. \tag{23}$$

This is easily checked in our case. Indeed, for $k \leq n$,

$$\mathbb{P}(\mathbf{srec} = k) = \frac{1}{n!}[z^k]\prod_{i=1}^{k}[z^i + i - 1]\prod_{u=k}^{n-1} u,$$

and, if k is large, by (23),

$$[z^k]\prod_{i=1}^{k}[z^i + i - 1] \sim k!\frac{e^{-\gamma}}{k}.$$

So

$$\mathbb{P}(\mathbf{srec} = k) \sim e^{-\gamma}\frac{1}{n!}(k-1)!\prod_{u=k}^{n-1} u = \frac{e^{-\gamma}}{n}.$$

For $k \leq n$ large enough, $\mathbb{P}(\mathbf{srec} = k)$ is asymptotically constant.

We have made a numerical comparison of $\mathbb{P}(\mathbf{srec} = k), n = 200, k = 1..3n$ with $\frac{e^{-\gamma}\rho(k/n)}{n}$. This is given in Figure 5 and is quite excellent.

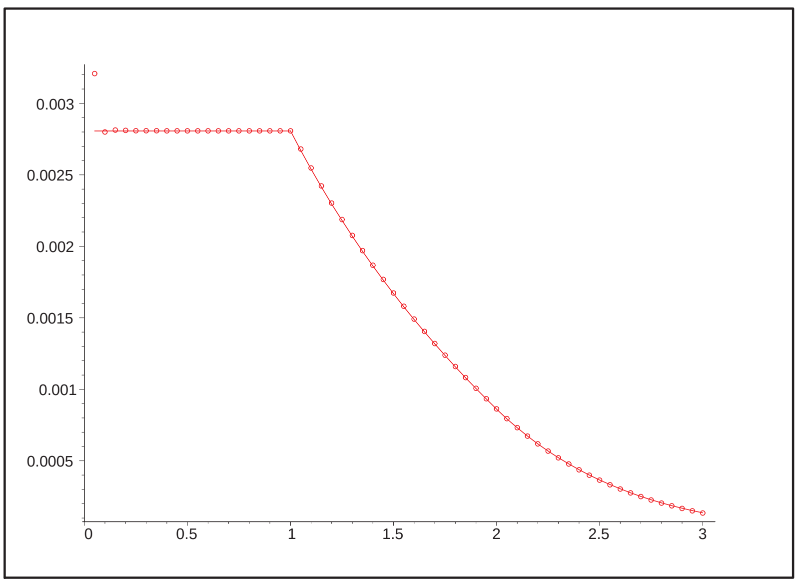

FIGURE 5. Comparison between $\mathbb{P}(\mathbf{srec} = k), n = 200, k = 1..3n$ (circle) as function of k/n and the asymptotics $\frac{e^{-\gamma}\rho(k/n)}{n}$ (line)

4. JUSTIFICATION OF THE INTEGRATION PROCEDURES USED IN THE LARGE DEVIATION SADDLE POINT METHOD.

We consider the large deviation $\ell = \frac{n(n+1)}{2}(1-y^2)$.

We will distinguish two domains: $\theta\tilde{i} > 1$ and $\theta\tilde{i} = o(1)$. (Recall that $\tilde{i} = n\alpha$). We are only interested here in rough asymptotics.

4.1. **The first domain $\theta\tilde{i} > 1$.** Let us first precisely compute a value $\tilde{k}$ such that $\frac{\tilde{z}^{\tilde{k}}}{\tilde{k}-1} = 1$. Set $l = \tilde{k}-1, l = \tilde{i}-\Delta$. We have

$$\begin{aligned}\tilde{z}^{\tilde{i}-\Delta+1} &= \tilde{i}-\Delta,\\ \tilde{z}^{-\Delta+1} &= 1-\Delta/\tilde{i},\\ \xi(-\Delta-1) &\sim \Delta/\tilde{i},\\ \Delta &\sim 1+1/L_1,\\ \tilde{k} &\sim \tilde{i}-1/L_1,\end{aligned}$$

where $L_1 := L+\ln(\alpha)$.

So we can choose the summation in Σ_3, (given by (16)) as

$$\begin{aligned}\Sigma_3(\theta) &= \sum_{k=1}^{\tilde{i}-1} \ln\left[(\tilde{z}e^{\mathbf{i}\theta})^k + k - 1\right]\\ &= \ln\left[\tilde{z}e^{\mathbf{i}\theta}\right] + \ln\left[1+(\tilde{z}e^{\mathbf{i}\theta})^2\right] + \sum_{k=3}^{\tilde{i}-1}\ln\left[(\tilde{z}e^{\mathbf{i}\theta})^k + k-1\right].\end{aligned}$$

The lower summation index 3 is justified below.

We are interested in $\Re(\Sigma_3(\theta)-\Sigma_3(0))$. This first leads, for $k=2$, to

$$\Re\left(\ln\left[1+(\tilde{z}e^{\mathbf{i}\theta})^2\right]\right) - \ln\left[1+\tilde{z}^2\right] \sim -\frac{2\tilde{z}^2}{(1+\tilde{z}^2)^2}\theta^2 \sim -\frac{1}{4}\theta^2, \tag{24}$$

and next, to

$$\Sigma_3^*(\theta) = \sum_{j=1}^{\infty}\frac{(-1)^{j+1}}{j}\sum_{k=3}^{\tilde{i}-1}\Re\left[\frac{\left(e^{\xi+\mathbf{i}\theta}\right)^{kj}}{(k-1)^j} - \frac{e^{\xi kj}}{(k-1)^j}\right] = \sum_{j=1}^{\infty}\frac{(-1)^{j+1}}{j}T_j,$$

where

$$T_j = \sum_{k=3}^{\tilde{i}-1}\frac{e^{\xi kj}}{(k-1)^j}\left[\cos(kj\theta)-1\right].$$

Note that, for $k=2$, $\frac{e^{\xi k}}{k-1} = e^{2\xi} > 1$ which should lead to an exponentially large value $e^{2\xi j}$ if $\xi j > 1$. But $\frac{e^{3\xi}}{2} < 1$ which is OK, hence our lower summation index 3.

Setting $k = \tilde{i}-u$,

$$T_j = \sum_{u=1}^{\tilde{i}-3}\frac{e^{\xi(\tilde{i}-u)j}}{\tilde{i}^j(1-u/\tilde{i}-1/\tilde{i})^j}\left[\cos(j(\tilde{i}-u)\theta)-1\right].$$

Using Euler-Maclaurin, setting $v = \xi u$,

$$T_j \sim \int_{\xi}^{L_1(1-3/\tilde{i})}\frac{e^{-jv}}{(1-v/L_1-1/\tilde{i})^j}\left[\cos(j\tilde{i}(1-v/L_1)\theta)-1\right]\frac{\tilde{i}}{L_1}dv. \tag{25}$$

The error terms are respectively given by

- If $u=1$, then

  $$\frac{e^{-j\xi}}{(1-1/\tilde{i}-1/\tilde{i})^j}\left[\cos(j\tilde{i}\theta)-1\right]$$

  is bounded above by

  $$2\frac{e^{-j\xi}}{(1-1/\tilde{i}-1/\tilde{i})^j} \sim 2e^{-j\xi}e^{2j/\tilde{i}} \sim 2e^{-j\xi}.$$

  This entails a maximum contribution to $\Sigma_3^*(\theta)$ bounded by

  $$\sum_{j=1}^{\infty}\frac{2}{j}e^{-j\xi} = -2\ln(1-e^{-\xi}) \sim 2e^{-\xi} \sim 2.$$

- If $u = \tilde{i} - \xi$, then
$$\frac{e^{-j\xi\tilde{i}}}{(1-(\tilde{i}-3)/\tilde{i}-1/\tilde{i})^j}\left[\cos(j3\theta)-1\right]$$
is bounded above by
$$2\frac{1}{\tilde{i}^y(2/\tilde{i})^j} = \frac{2}{2^j}.$$
This entails a maximum contribution to $\Sigma_3^*(\theta)$ bounded by
$$\sum_{j=1}^{\infty}\frac{2}{j}\frac{1}{2^j} = -2\ln(1-1/2) = 1.39\ldots.$$

If $\tilde{i}\theta \succ 1$, the strongly oscillating character of $\cos(j\tilde{i}(1-v/L_1\theta)$ leads asymptotically, when integrated, to 0. Indeed,
$$\int_0^{\delta}\cos(\beta v)dv = \frac{\sin(\beta\delta)}{\beta} = o(1)$$
for large β.

Now the function
$$F(v) := \frac{e^{-v}}{(1-v/L_1-1/\tilde{i})}$$
is such that $F(\xi) = e^{-\xi}/(1-2/\tilde{i}) \sim 1, F(L_1-3\xi) = \frac{e^{3\xi}}{2} < 1$, $F(v)$ decreases from $v=\xi$ to a unique minimum at $v \sim L_1-1$ where it has the value $\sim eL_1/\tilde{i} = e\xi$ and increases to $v = L_1-3\xi$ where it has the value $\sim \frac{1}{2}$..

- Now we consider
$$\begin{aligned}
&\int_{\xi}^{L_1-3\xi}\frac{e^{-jv}}{(1-v/L_1-1/\tilde{i})^j}dv, \text{ and setting } v=\xi+s\\
&= \frac{e^{-j\xi}}{(1-2/\tilde{i})^j}\int_0^{L_1-4\xi}\frac{e^{-js}}{(1-\frac{s}{L_1}\frac{1}{1-2/\tilde{i}})^j}ds, \text{ and setting } js=t\\
&\sim e^{-j(\xi-2/\tilde{i})}\int_0^{(L_1-4\xi)j}\frac{e^{-t}}{(1-\frac{t}{jL_1}\frac{1}{1-2/\tilde{i}})^j}\frac{dt}{j}\\
&\sim \frac{e^{-j\xi}}{j}\int_0^{(L_1-4\xi)j}\frac{e^{-t}}{(1-\frac{t}{jL_1}\frac{1}{1-2/\tilde{i}})^j}dt.
\end{aligned}$$

- We first estimate the integral
$$I_1 = \int_0^{(L_1-1)j}\frac{e^{-t}}{(1-\frac{t}{jL_1}\frac{1}{1-2/\tilde{i}})^j}dt \sim \int_0^{(L_1-1)j}\frac{e^{-t}}{(1-\frac{t}{jL_1})^j}dt \sim \int_0^{(L_1-1)j}e^{t/L_1}dt \sim \int_0^{(L_1-1)j}e^{-t}dt \sim 1.$$

- We next consider the integral
$$I_2 = \int_{(L_1-1)j}^{(L_1-4\xi)j}\frac{e^{-t}}{(1-\frac{t}{jL_1}\frac{1}{1-2/\tilde{i}})^j}dt,$$
and setting $t = (L_1-4\xi)j-u, A = \frac{i}{2L_1} = \frac{1}{2\xi} \succ 1$,
$$I_2 \sim \int_0^j\frac{(1/i)^je^u}{(\frac{2}{i}+\frac{u}{jL_1})^j}du = \frac{1}{2^j}\int_0^j\frac{e^u}{(1+\frac{uA}{j})^j}du.$$
This last integral can itself be decomposed into two integrals, setting $u^* := \frac{j}{A} = o(j)$

First of all, we have

$$I_{21} = \frac{1}{2^j}\int_0^{u^*} \frac{e^u}{(1+\frac{uA}{j})^j}du \sim \frac{1}{2^j}\int_0^{u^*} \frac{e^u}{e^{uA}}du = \frac{1}{2^j}\frac{1}{A}\left[1-e^{-j}\right].$$

Next we analyze

$$I_{22} = \frac{1}{2^j}\int_{u^*}^{j} \frac{e^u}{(1+\frac{uA}{j})^j}du = \frac{u^{*j}}{2^j}\int_{u^*}^{j} \frac{e^u}{(u+u^*)^j}du.$$

For $u=u^*$, the integrand gives

$$\frac{e^{u^*}}{(2u^*)^j},$$

and the derivative is given by

$$\sim -\frac{A}{2}\frac{e^{u^*}}{(2u^*)^j}.$$

The integrand minimum is attained at $u = j(1-\frac{1}{A})$ and is given by

$$\frac{e^j}{(j+u^*)^j} \sim \frac{e^j}{j^j}.$$

Multiplying by $\frac{u^{*j}}{2^j}$, this leads to $(e\xi)^j$, as expected.

Some numerical experiments show that the integrand is asymptotically (for large A) exponentially decreasing in the neighbourhood of u^*. This should lead to

$$\frac{e^{u^*}}{(2u^*)^j}e^{-\frac{A}{2}(u-u^*)}.$$

Setting $u = wu^*$, $w \in [1,A]$, we must compare

$$e^{u^*}\frac{e^{-(w-1)j/2}}{(2u^*)^j}$$

with

$$\frac{e^{wu^*}}{u^{*j}(1+w)^j},$$

or

$$j\left[\frac{1}{A}-\frac{w-1}{2}-\ln(2)\right]$$

with

$$j\left[\frac{w}{A}-\ln(1+w)\right].$$

For $w=1$, we have equality. For $w=1+\varepsilon, \varepsilon = o(1)$, the first term of the expansion coincides. For $w=2$, we have respectively $\sim -j1.2\ldots$ and $\sim -j1.1\ldots$. Of course, both expressions become negatively large for large j. But even for small values of j, the two integrals

$$\int_1^A \exp\left(j\left[\frac{1}{A}-\frac{w-1}{2}-\ln(2)\right]\right)dw$$

and

$$\int_1^A \exp\left(j\left[\frac{w}{A}-\ln(1+w)\right]\right)dw$$

give rather close values: for $j=3, A=100$, we obtain respectively $0.09\ldots, 012\ldots$, for $j=4, A=100$, we obtain $0.033\ldots, 0.045\ldots$,for $j=5, A=100$, we obtain $0.013\ldots, 0.017\ldots$. So we can use the rough approximation

$$I_{22} \sim \frac{u^{*j}}{2^j}\int_{u^*}^{j} \frac{e^{u^*}}{(2u^*)^j}e^{-\frac{A}{2}(u-u^*)}du = \frac{e^{u^*}}{4^j}\frac{2}{A}\left[1-e^{-jA/2}\right] \sim \left(\frac{e^{1/A}}{4}\right)^j\frac{2}{A}.$$

Using I_1, I_{21}, I_{22} this finally gives

$$T_j \sim -\frac{\tilde{i}}{L_1}\frac{e^{-j\xi}}{j}\left[1+\frac{1}{2^j}\frac{1}{A}\left[1-e^{-j}\right]+\left(\frac{e^{1/A}}{4}\right)^j\frac{2}{A}\right]$$

and

$$\Sigma_3^*(\theta)=\sum_{j=1}^{\infty}\frac{(-1)^{j+1}}{j}T_j \sim \frac{\tilde{i}}{L_1}\left[polylog(2,1)+\frac{1}{A}polylog(1,1/2)-\frac{1}{A}polylog(1,e^{-1}/2)+\frac{2}{A}polylog(1,1/4)\right]$$

(26)

$$\sim -\frac{\pi^2}{12}\frac{\tilde{i}}{L_1}.$$

For instance, for $n=10^4, \theta=2, \alpha=1/2$, we have $\Re[\Sigma_3(\theta)-\Sigma_3(0)]=-570$ and $\Sigma_3^*(\theta)=-482$, which is quite satisfactory. The analysis of $\Sigma_4^*(\theta)$ given by

$$\Sigma_4^*(\theta)=\sum_{j=1}^{\infty}\frac{(-1)^{j+1}}{j}\sum_{k=\tilde{i}}^{n}(k-1)^j e^{-\xi kj}\left[\cos(kj\theta)-1\right]$$

is quite analogous, we omit the details. The corrections due to (24) are negligible.

4.2. **The second domain $\theta\tilde{i}=o(1)$.** Set for instance $\theta=\frac{1}{\tilde{i}\varphi(n)}$ where $\varphi(n)$ is an increasing function of n. We split the summation on j into three parts: $\Sigma_{31}^*(\theta)=\sum_{j=1}^{\tilde{j}/2-1}$, $\Sigma_{32}^*(\theta)=\sum_{j=\tilde{j}/2}^{\tilde{j}}$ and $\Sigma_{33}^*(\theta)=\sum_{j=\tilde{j}+1}^{\infty}$, where $\tilde{j}:=\frac{\pi/2}{\tilde{i}\theta}=\pi/2\varphi(n)$. We assume that $\tilde{j}$ is an even integer and $\tilde{j}/2$ is an odd integer (or, w.l.g. the closest value to $\tilde{j}$ satisfying these requirements).

- First subdomain $j\le\tilde{j}/2-1$

  As $j\tilde{i}\theta<\pi/2$, we can use the first order approximation $-\left[j\tilde{i}(1-v/L_1)\theta\right]^2/2$ for $\left[\cos(j\tilde{i}(1-v/L_1)\theta)-1\right.$ We have the asymptotics

  $$T_j \sim -\int_{\xi}^{L_1(1-3/\tilde{i})}\frac{e^{-jv}}{(1-v/L_1-1/\tilde{i})^j}\left[j\tilde{i}(1-v/L_1)\theta\right]^2/2\frac{\tilde{i}}{L_1}dv.$$

  Also, for j sufficiently large, due to the presence of e^{-jv}, we can neglect the contribution of v/L_1. Proceeding as above,

  $$T_j \sim -e^{-j\xi}j\frac{\tilde{i}^3\theta^2}{2L_1}.$$

  This gives

  $$\Sigma_{31}^*(\theta)\sim -\frac{e^{-\xi}-e^{-\tilde{j}/2\xi}}{e^{\xi}+1}\frac{\tilde{i}^3\theta^2}{2L_1}.$$

  (1) If $\tilde{j}\xi=o(1)$ i.e. $\varphi(n)=o(n)$ then

  $$\frac{e^{-\xi}-e^{-\tilde{j}/2\xi}}{e^{\xi}+1}\sim\tilde{j}\xi/4=o(1).$$

  (2) If $\tilde{j}\xi\succ 1$ i.e. $\varphi(n)\succ n$ then

  $$\frac{e^{-\xi}-e^{-\tilde{j}/2\xi}}{e^{\xi}+1}\sim 1/2.$$

- Second subdomain $\tilde{j}/2\le j\le\tilde{j}$ Now we have

  $$T_j \sim -\frac{e^{-\xi j}}{j}\left[1-\cos\left(\frac{j}{\tilde{j}}\frac{\pi}{2}\right)\right]\frac{\tilde{i}}{L_1}$$

and

$$\Sigma^*_{32}(\theta) \sim -\sum_{j=\tilde{j}/2}^{\tilde{j}} \frac{(-1)^{j+1}}{j}\frac{e^{-\xi j}}{j}\left[1-\cos\left(\frac{j}{\tilde{j}}\frac{\pi}{2}\right)\right]\frac{\tilde{i}}{L_1}.$$

For s odd and u even, we have

$$\sum_{j=s}^{u}\frac{(-1)^{j+1}}{j^2}f(j) = \sum_{j=s}^{u}\frac{f(j)}{j^2} - \frac{1}{2}\sum_{v=(s+1)/2}^{u/2}\frac{f(2v)}{v^2}.$$

hence

$$\Sigma^*_{32}(\theta) \sim -\frac{\tilde{i}}{L_1}\left[\sum_{j=\tilde{j}/2}^{\tilde{j}}\frac{e^{-\xi j}}{j^2}\left[1-\cos\left(\frac{j}{\tilde{j}}\frac{\pi}{2}\right)\right] - \frac{1}{2}\sum_{j=\tilde{j}/4+1/2}^{\tilde{j}/2}\frac{e^{-2\xi j}}{j^2}\left[1-\cos\left(2\frac{j}{\tilde{j}}\frac{\pi}{2}\right)\right]\right].$$

(1) If $\tilde{j}\xi = o(1)$ i.e. $\varphi(n) = o(n)$ then, setting $j = \tilde{j}/2+u, \varepsilon = \xi\tilde{j}/2 = o(1)$, the first summation leads to

$$\begin{aligned}
&e^{-\xi\tilde{j}/2}\int_{u=0}^{\tilde{j}/2}\frac{e^{-\xi u}}{(\tilde{j}/2+u)^2}\left[1-\cos\left[(1/2+u/\tilde{j})\pi/2\right]\right]du \\
&\sim \frac{e^{-\varepsilon}}{(\tilde{j}/2)^2}\int_{v=0}^{\varepsilon}\frac{e^{-v}}{(1+v/\varepsilon)^2}\left[1-\cos\left[(1/2+v/(2\varepsilon))\pi/2\right]\right]\frac{dv}{\xi} \\
&\sim \frac{e^{-\varepsilon}}{(\tilde{j}/2)^2}\varepsilon\int_{w=0}^{1}\frac{e^{-w\varepsilon}}{(1+w)^2}\left[1-\cos\left[(1/2+w/2)\pi/2\right]\right]\frac{dw}{\xi} \\
&\sim C_1\frac{e^{-\varepsilon}}{\tilde{j}/2}, \text{ with} \\
&C_1 = \int_{w=0}^{1}\frac{1}{(1+w)^2}\left[1-\cos\left[(1/2+w/2)\pi/2\right]\right]dw = 0.27\ldots.
\end{aligned}$$

Setting $j = \tilde{j}/4+1/2+u, \eta = \xi(\tilde{j}/2+1) = \varepsilon+\xi$, the second summation leads to

$$\begin{aligned}
&e^{-\xi(\tilde{j}/4+1/2)2}\int_{u=0}^{\tilde{j}/4-1/2}\frac{e^{-2\xi u}}{(\tilde{j}/4+1/2+u)^2}\left[1-\cos\left[(1/2+1/\tilde{j}+2u/\tilde{j})\pi/2\right]\right]du \\
&\sim \frac{e^{-\eta}}{(\tilde{j}/4+1/2)^2}\int_{v=0}^{\eta-2\xi}\frac{e^{-v}}{(1+v/\eta)^2}\left[1-\cos\left[(1/2+1/\tilde{j}+v/(\xi\tilde{j}))\pi/2\right]\right]\frac{dv}{2\xi} \\
&\sim \frac{e^{-\eta}}{(\tilde{j}/4+1/2)^2}\eta\int_{w=0}^{1-2\xi/\eta}\frac{e^{-w\eta}}{(1+w)^2}\left[1-\cos\left[(1/2+1/\tilde{j}+w\eta/(\xi\tilde{j}))\pi/2\right]\right]\frac{dw}{2\xi} \\
&\sim \frac{e^{-\eta}}{\tilde{j}/4+1/2}\int_{w=0}^{1-1/(\tilde{j}/4+1/2)}\frac{e^{-w\eta}}{(1+w)^2}\left[1-\cos\left[(1/2+1/\tilde{j}+w(1/2+1/\tilde{j}))\pi/2\right]\right]dw \\
&\sim \frac{e^{-\eta}}{\tilde{j}/4+1/2}\left[C_1+C_2/\tilde{j}\right] \text{ with } C_2 = -0.04\ldots.
\end{aligned}$$

So, finally,

$$\begin{aligned}
\Sigma^*_{32}(\theta) &\sim -\frac{\tilde{i}}{L_1}\left[e^{-\varepsilon}C_1\left[\frac{1}{\tilde{j}/2}-\frac{e^{-\xi}}{\tilde{j}/2+1}\right]-\frac{1}{2}C_2\frac{e^{-\eta}}{\tilde{j}/4+1/2}\frac{1}{\tilde{j}}\right] \\
&\sim -\frac{\tilde{i}}{L_1}\left[e^{-\varepsilon}C_1\frac{1}{(\tilde{j}/2)^2}-2C_2\frac{e^{-\eta}}{\tilde{j}^2}\right] \text{ as } \frac{1}{\tilde{j}^2} \succ \left(\frac{\xi}{\tilde{j}}\right), \\
&\sim -\frac{\tilde{i}}{L_1}\frac{1}{\tilde{j}^2}\left[4C_1-2C_2\right] = -1.17\ldots\frac{\tilde{i}^3\theta^2}{(\pi/2)^2L_1}.
\end{aligned}$$

(2) If $\tilde{j}\xi > 1$ i.e. $\varphi(n) > n$ then setting $M = \tilde{j}\xi/2 > 1$,the first summation leads to

$$\sim \frac{e^{-M}}{(\tilde{j}/2)^2}\int_{v=0}^{M}\frac{e^{-v}}{(1+v/M)^2}\left[1-\cos\left[(1/2+v/(2M))\pi/2\right]\right]\frac{dv}{\xi}$$
$$\sim \frac{e^{-M}}{(\tilde{j}/2)^2}\int_{v=0}^{M}e^{-v}e^{-2v/M}\left[1-\cos\left[(1/2+v/(2M))\pi/2\right]\right]\frac{dv}{\xi}$$
$$\sim \frac{e^{-M}}{(\tilde{j}/2)^2}\frac{1}{\xi}\left[1-\cos(\pi/4)\right].$$

Setting $N = (\xi(\tilde{j}/2+1) = M+\xi$, the second summation leads to

$$\sim \frac{e^{-N}}{(\tilde{j}/4+1/2)^2}\int_{v=0}^{N-2\xi}\frac{e^{-v}}{(1+v/N)^2}\left[1-\cos\left[(1/2+1/\tilde{j}+v/(2N-2\xi)\pi/2\right]\right]\frac{dv}{2\xi}$$
$$\sim \frac{e^{-N}}{(\tilde{j}/4+1/2)^2}\frac{1}{2\xi}\left[1-\cos(\pi/4)\right]$$

So finally

$$\Sigma_{32}^*(\theta) \sim -\frac{\tilde{i}}{L_1}\frac{e^{-M}}{(\tilde{j}/2)^2}\frac{1}{\xi}\left[1-\cos(\pi/4)\right]\left[1-\frac{e^{-\xi}}{(1+2/\tilde{j})^2}\right] \sim -\frac{\tilde{i}}{L_1}\frac{e^{-M}}{(\tilde{j}/2)^2}\left[1-\cos(\pi/4)\right].$$

- Third subdomain $j > \tilde{j}$

  We use (25), starting with $T_{\tilde{j}+1}$. Of course $\cos((\tilde{j}+1)\tilde{i}(1-v/L_1)\theta) \sim \cos(\pi/2(1-v/L_1))$ is slowly oscillating, but, by the presence of e^{-jv}, this is asymptotically equivalent to $\cos(\pi/2) = 0$, and, when j gets larger, the oscillating character of $\cos((\tilde{j}+1)\tilde{i}(1-v/L_1)\theta))$ leads also asymptotically to 0. So

$$T_j \sim -\frac{\tilde{i}}{L_1 j}e^{-j\xi},$$

  and

$$\Sigma_{33}^*(\theta) \sim -\sum_{j=\tilde{j}+1}^{\infty}\frac{(-1)^{j+1}}{j^2}e^{-j\xi}\frac{\tilde{i}}{L_1}.$$

  The analysis is somewhat simpler than in the previous subdomain. Let $x < 1, s$ odd. We start from

(27) $$\sum_{j=s}^{\infty}\frac{(-1)^{j+1}}{j^2}x^j = \sum_{j=s}^{\infty}\frac{1}{j^2}x^j - \frac{1}{2}\sum_{j=(s+1)/2}^{\infty}\frac{1}{j^2}x^{2j}.$$

  For large s, setting $j = s+u, x = e^{-\xi}$,

$$\sum_{j=s}^{\infty}\frac{1}{j^2}e^{-\xi j} \sim F(s,\xi),$$

$$F(s,\xi) := e^{-\xi s}\int_0^{\infty}\frac{e^{-\xi u}}{(s+u)^2}du = \frac{e^{-\xi s}}{s^2}\int_0^{\infty}\frac{e^{-v}}{(1+v/(\xi s))^2}\frac{dv}{\xi}.$$

(1) If $\tilde{j}\xi = o(1)$ i.e. $\varphi(n) = o(n)$ then, setting $\varepsilon = (\tilde{j}+1)\xi = o(1)$,

$$F(\tilde{j}+1,\xi) \sim \frac{e^{-\varepsilon}}{(\tilde{j}+1)^2}\int_0^{\infty}\frac{e^{-v}}{(1+v/\varepsilon)^2}\frac{dv}{\xi}.$$

Now

$$\int_0^{\infty}\frac{e^{-v}}{(1+v/\varepsilon)^2}dv = \int_0^{\varepsilon}\frac{e^{-v}}{(1+v/\varepsilon)^2}dv + \int_{\varepsilon}^{\infty}\frac{e^{-v}}{(1+v/\varepsilon)^2}dv \sim \varepsilon\int_0^{1}\frac{e^{-u\varepsilon}}{(1+u)^2}du + \varepsilon\int_1^{\infty}\frac{e^{-u\varepsilon}}{(1+u)^2}du \sim \varepsilon.$$

Hence

$$F(\tilde{j}+1,\xi) \sim \frac{e^{-\varepsilon}}{(\tilde{j}+1)^2}\frac{1}{\xi}\varepsilon = \frac{1}{(\tilde{j}+1)}e^{-(\tilde{j}+1)\xi}.$$

Using now (27), we derive

$$F(\tilde{j}+1,\xi)-\frac{1}{2}F((\tilde{j}+2)/2,2\xi)\sim e^{-(\tilde{j}+1)\xi}\left[\frac{1}{\tilde{j}+1}-\frac{e^{-\xi}}{\tilde{j}+2}\right]\sim e^{-\varepsilon}\frac{1}{(\tilde{j}+1)^2}\sim\frac{1}{\tilde{j}^2},$$

as

$$\frac{1}{\tilde{j}^2}\succ\left(\frac{\xi}{\tilde{j}}\right).$$

Then

$$\Sigma_{33}^*(\theta)\sim-\frac{1}{\tilde{j}^2}\frac{\tilde{i}}{L_1}=-\frac{\tilde{i}^3\theta^2}{(\pi/2)^2L_1}.$$

(2) If $\tilde{j}\xi\succ 1$ i.e. $\varphi(n)\succ n$ then setting $M=(\tilde{j}+1)\xi\succ 1$,

$$F(\tilde{j}+1,\xi)\sim\frac{e^{-M}}{(\tilde{j}+1)^2}\int_0^\infty\frac{e^{-v}}{(1+v/M)^2}\frac{dv}{\xi}.$$

Now

$$\int_0^\infty\frac{e^{-v}}{(1+v/M)^2}dv=\int_0^M\frac{e^{-v}}{(1+v/M)^2}dv+\int_M^\infty\frac{e^{-v}}{(1+v/M)^2}dv$$
$$\sim\int_0^M e^{-v}e^{-2v/M}dv+M^2\int_M^\infty\frac{e^{-v}}{v^2(1+M/v)^2}dv\sim 1+\mathcal{O}(e^{-M})\sim 1.$$

Hence

$$F(\tilde{j}+1,\xi)\sim\frac{e^{-M}}{(\tilde{j}+1)^2}\frac{1}{\xi}.$$

Using again (27), we derive

$$F(\tilde{j}+1,\xi)-\frac{1}{2}F((\tilde{j}+2)/2,2\xi)\sim\frac{e^{-(\tilde{j}+1)\xi}}{(\tilde{j}+1)^2\xi}\left[1-\frac{e^{-\xi}}{(1+1/(\tilde{j}+1))^2}\right]\sim e^{-M}\frac{1}{(\tilde{j}+1)^2}\sim e^{-M}\frac{1}{\tilde{j}^2},$$

as

$$\frac{2}{\tilde{j}}=o(\xi).$$

Then

$$\Sigma_{33}^*(\theta)\sim-\frac{1}{\tilde{j}^2}e^{-M}\frac{\tilde{i}}{L_1}=-\frac{\tilde{i}^3\theta^2}{(\pi/2)^2L_1}e^{-M}.$$

- So finally
  (1) $\Sigma_3^*(\theta)=\Sigma_{31}^*(\theta)+\Sigma_{32}^*(\theta)+\Sigma_{33}^*(\theta)\sim-\frac{\tilde{i}^3\theta^2}{(\pi/2)^2L_1}(1+4C_1-2C_2+\mathcal{O}(\varepsilon))$ if $\varphi(n)=o(n)$,
  (2) $\Sigma_3^*(\theta)=\Sigma_{31}^*(\theta)+\Sigma_{32}^*(\theta)+\Sigma_{33}^*(\theta)\sim-\frac{\tilde{i}^3\theta^2}{4L_1}(1+\mathcal{O}(e^{-M})$ if $\varphi(n)\succ n$.

  This is compatible with (19).

  For instance, for $n=10^3,\theta=2.10^{-5},\alpha=1/2,\varphi(n)=o(n)$, we have $\Re[\Sigma_3(\theta)-\Sigma_3(0)]=-0.0014\ldots$ and $\Sigma_3^*(\theta)=-0.0069\ldots$, which has the correct order of magnitude. Of course, we used rough asymptotics. For $n=10^3,\theta=2.10^{-7},\alpha=1/2,\varphi(n)\succ n$, we have $\Re[\Sigma_3(\theta)-\Sigma_3(0)]=-0.00000014\ldots$ and $\Sigma_3^*(\theta)=-0.00000020\ldots$, which is again satisfactory. Again, the corrections due to (24) are negligible. Note that the case $\theta=1/\tilde{i}$ is compatible with (26). Also the analysis of $\Sigma_4^*(\theta)$ is quite analogous.

5. Conclusion.

Using the symbolic computer system Maple, we have obtained some asymptotic expressions for the sum of positions of records in random permutations in central and non-central regions. The saddle point method proved again to be a powerful tool in our expansions computation.

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