Asymptotics of the Stirling numbers of the first kind revisited:
A saddle point approach

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Abstract

Using the saddle point method, we obtain from the generating function of the Stirling numbers of the first kind $\left[\begin{array}{c} n \\ j \end{array}\right]$ and Cauchy's integral formula, asymptotic results in central and non-central regions. In the central region, we revisit the celebrated Goncharov theorem with more precision. In the region $j = n - n^\alpha, \; \alpha > 1/2$, we analyze the dependence of $\left[\begin{array}{c} n \\ j \end{array}\right]$ on $\alpha$.

Keywords: Stirling numbers, saddle point method.

1 Introduction

Let $\left[\begin{array}{c} n \\ j \end{array}\right]$ be the Stirling number of the first kind (unsigned version). Their generating function is given by

$$\phi_n(z) = \prod_{i=0}^{n-1} (z + i) = \frac{\Gamma(z + n)}{\Gamma(z)}, \; \phi_n(1) = n!.$$ 

An asymptotic expansion for $j = O(1)$ is given in Wilf [14], which has been extended to the range $j = O(\ln n)$ by Hwang [6]. The generalized Sirling numbers have been considered by Tsylova [13] and Chelluri et al. [2]. The $q-$Stirling numbers are studied in Kyriakoussis and Vamvakari [9].

In Sec.2, we revisit the asymptotic expansions in the central region and in Sec.3, we analyze the non-central region $j = n - n^\alpha, \; \alpha > 1/2$. We use Cauchy’s integral formula and the saddle point method.

2 Central region

Consider

$$J_n(j) := \left[\begin{array}{c} n \\ j \end{array}\right]$$

as a random variable. The mean and variance are given by

$$M := E(J_n) = \sum_{i=0}^{n-1} \frac{1}{i+1} = H_n = \psi(n + 1) + \gamma,$$

$$\sigma^2 := V(J_n) = \sum_{i=0}^{n-1} \frac{i}{(1+i)^2} = \psi(1, n + 1) + \psi(n + 1) - \frac{\pi^2}{6} + \gamma,$$

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and
\[
M \sim \ln(n) + \gamma + \frac{1}{2n} + \mathcal{O}\left(\frac{1}{n^2}\right),
\]
\[
\sigma^2 \sim \ln(n) - \frac{\pi^2}{6} + \gamma + \mathcal{O}\left(\frac{1}{n^2}\right).
\]

It is convenient to set
\[
A := \ln(n) - \frac{\pi^2}{6} + \gamma = \ln\left(ne^{\gamma-\pi^2/6}\right),
\]
and to consider all our next asymptotics \((n \to \infty)\) as functions of \(A\). Of course, all asymptotics can be reformulated in terms of \(\ln(n)\).

We have
\[
M \sim A + \frac{\pi^2}{6} + \mathcal{O}\left(\frac{1}{n}\right),
\]
\[
\sigma^2 \sim A + \mathcal{O}\left(\frac{1}{n}\right).
\]

A celebrated theorem of Goncharov says that
\[
J_n(j) \sim \mathcal{N}\left(\frac{j - M}{\sigma}\right),
\]
where \(\mathcal{N}\) is the Gaussian distribution, with a rate of convergence \(\mathcal{O}(1/\sqrt{\ln(n)})\). This can also be deduced from the Quasi-Power theorem of Hwang [7],[8].

In this Section, we want to obtain a more precise local limit theorem for \(J_n(j)\) in terms of \(x := \frac{j - M}{\sigma}\) and \(A\).

By Cauchy’s theorem,
\[
J_n(j) = \frac{1}{2\pi i} \int_{\Omega} \phi_n(z) \frac{dz}{z^{j+1}}
\]
\[
= \frac{1}{2\pi i} \int_{\Omega} e^{S(z)} dz,
\]
where \(\Omega\) is inside the analyticity domain of the integrand and encircles the origin and
\[
S(z) = S_1(z) + S_2(z), \quad S_1(z) = \sum_{i=0}^{n-1} \ln(z + i) - \ln(n!), \quad S_2(z) = -(j + 1) \ln(z).
\]

Set
\[
S^{(i)} := \frac{d^i S}{dz^i}.
\]
These derivatives can be expressed in terms of \(\psi(k, z + n)\) and \(\psi(k, z)\).

We will use the Saddle point method (for a good introduction to this method, see Flajolet and Sedgewick [3, ch.VIII]). First we must find the solution of
\[
S^{(1)}(\hat{z}) = 0 \quad \text{(1)}
\]
with smallest module.

Set \(\hat{z} := z^* - \varepsilon\), where, here, it is easy to check that \(z^* = 1\). Set \(j = M + x\sigma\) and \(B := \sqrt{A}\) to simplify the expressions.

This leads, to first order (keeping only the \(\varepsilon\) term in (1)), to
\[
\varepsilon := -\frac{x}{B} + \frac{x^2 - 1}{B^2} + \mathcal{O}\left(\frac{1}{B^3}\right) + \frac{1}{n} \left(\frac{3x}{4B^3} + \mathcal{O}\left(\frac{1}{B^4}\right)\right) + \mathcal{O}\left(\frac{1}{n^2B^4}\right).
\]
This shows that, asymptotically, \( \varepsilon \) is given by a Laurent series of powers of \( n^{-1} \), where each coefficient is given by a Laurent series of powers of \( B^{-1} \). To obtain more precision, we set again \( j = M + x \sigma \), expand in powers of \( n^{-1} \), and equate each coefficient to 0. Note that we will need the \( 1/n \) term of \( \varepsilon \) later on. This leads to

\[
\varepsilon = \frac{-x}{B} - \frac{1}{B^2} + \frac{0}{B^3} + \mathcal{O} \left( \frac{1}{B^4} \right) + \frac{1}{n} \left( \frac{3x}{4B^3} + \frac{x^2 + 3/2}{B^4} + \mathcal{O} \left( \frac{1}{B^6} \right) \right) + \mathcal{O} \left( \frac{1}{n^2B^4} \right).
\]

We have, with \( \tilde{z} := z^* - \varepsilon = 1 - \varepsilon \),

\[
J_n(j) = \frac{1}{2\pi i} \int_\Omega \exp \left[ S(\tilde{z}) + S^{(2)}(\tilde{z})(z - \tilde{z})^2/2! + \sum_{l=3}^{\infty} S^{(l)}(\tilde{z})(z - \tilde{z})^l/l! \right] dz.
\]

Note that the linear term vanishes. Set \( z = \tilde{z} + i\tau \). This gives

\[
J_n(j) = \frac{1}{2\pi} \exp[S(\tilde{z})] \int_{-\infty}^{\infty} \exp \left[ S^{(2)}(\tilde{z})(i\tau)^2/2! + \sum_{l=3}^{\infty} S^{(l)}(\tilde{z})(i\tau)^l/l! \right] d\tau. \tag{2}
\]

The justification of the integration procedure is given in the Appendix. Let us first analyze \( S(\tilde{z}) \). We obtain (we see now why we need the \( 1/n \) term of \( \varepsilon \)): there is a summation \( \sum_{l=0}^{n-1} \) in \( S_1(\tilde{z}) \)

\[
S(\tilde{z}) = -x^2/2 + \frac{x^3/6 - x}{B} + \frac{-x^4/12 + x^2/2 - 1/2}{B^2}
+ \frac{-x^3/3 + x^5/20 + x^2 - \pi^2 x^3/18 + \zeta(3)x^3/3}{B^3} + \mathcal{O} \left( \frac{1}{B^4} \right) + \mathcal{O} \left( \frac{1}{nB^3} \right).
\]

Also, (here and in the following, we provide only a few terms but Maple knows more).

\[
S^{(2)}(\tilde{z}) = B^2 - Bx - 1 + x^2 + \ldots,
\]

\[
S^{(3)}(\tilde{z}) = -2B^2 + 4Bx - \pi^2/3 + 2\zeta(3) - 6x^2 + 4 + \ldots,
\]

\[
S^{(4)}(\tilde{z}) = 6B^2 - 18Bx + 36x^2 - 18 + \pi^2 - \pi^4/15 + \ldots,
\]

\[
S^{(l)}(\tilde{z}) = \mathcal{O} \left( B^2 \right), l \geq 5.
\]

We need these many terms in the following. Note that, with \( z = \tilde{z}e^{i\theta} \), this leads to

\[
S^{(2)}(\tilde{z})(z - \tilde{z})^2/2 \sim \frac{-1}{2\ln(n)}\theta^2. \tag{3}
\]

We can now compute (2), for instance by using the classical trick of setting

\[
S^{(2)}(\tilde{z})(i\tau)^2/2! + \sum_{l=3}^{\infty} S^{(l)}(\tilde{z})(i\tau)^l/l! = -u^2/2.
\]

Computing \( \tau \) as a truncated series in \( u \), this gives, by inversion,

\[
\tau = \left[ u(1 + x/(2B) + \ldots) + u^2(1/(3B) + \ldots) + u^3(-1/(36B^2) + \ldots) \right] / B + \ldots
\]

Setting \( d\tau = du/B \), expanding w.r.t. \( B \) and integrating on \( [u = -\infty, \infty] \), this gives

\[
\frac{1}{\sqrt{2\pi B}} \int \left[ 1 + \frac{x}{2B} + \frac{5/12 - x^2/8}{B^2} + \frac{x(8\pi^2 - 10 - 93x^2 - 48\zeta(3))}{48B^3} + \ldots \right].
\]

Finally (2) leads to

\[
J_n(j) \sim \frac{1}{\sqrt{2\pi B}} e^{-x^2/2} \cdot \exp \left[ \frac{x^3/6 - x}{B} + \frac{-x^4/12 + x^2/2 - 1/2}{B^2} + \frac{-x^3/3 + x^5/20 + x/2 - \pi^2 x^3/18 + \zeta(3)x^3/3}{B^3} + \ldots \right]
\cdot \left[ 1 + \frac{x}{2B} + \frac{5/12 - x^2/8}{B^2} + \frac{x(8\pi^2 - 10 - 93x^2 - 48\zeta(3))}{48B^3} + \ldots \right],
\]

\[ \text{3} \]
or

\[ J_n(j) \sim R_1, \]

\[ R_1 = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}. \]

\[ \cdot \left[ 1 + \frac{x^3/6 - x/2}{B} + \frac{3x^2/8 - x^4/6 - 1/12 + x^6/72}{B^2} \right. \]
\[ + \frac{-\pi x^3/18 + 37x^5/240 - 355x^3/144 + x/8 - x^7/48 + x^9/1296 + \pi^2 x/6 - \zeta(3)x + \zeta(3)x^3/3}{B^3} \ldots \].

For \( n = 3000 \), a comparison between \( J_n(j) \) and \( \frac{1}{\sqrt{2\pi\sigma}} \exp \left[ - \left( \frac{i-M}{\sigma} \right)^2 / 2 \right] \) is given in Figure 1.

![Figure 1: Comparison between \( J_n(j) \) and \( \frac{1}{\sqrt{2\pi\sigma}} \exp \left[ - \left( \frac{i-M}{\sigma} \right)^2 / 2 \right] \), \( n = 3000 \)](image)

Of course, only few values of \( j \) are significant and also the quality of the Gaussian is low, all asymptotic expressions depend actually on powers of \( A^{-1} \), but \( A \) is not large.

A comparison of \( J_n(j) / \left[ \frac{1}{\sqrt{2\pi\sigma}} \exp \left[ - \left( \frac{i-M}{\sigma} \right)^2 / 2 \right] \right] \) with \( J_n(j)/R_1 \), with 2 terms in \( R_1 \), is given in Figure 2.

The precision of \( R_1 \) is of order \( 10^{-2} \). Using 3 terms in \( R_1 \) leads to a less good result: \( A \) is not large enough to take advantage of the \( A^{-3/2} \) term: \( A = 6.94 \) here, we deal with asymptotic series, not necessarily convergent ones. More terms can be computed in \( R_1 \) (which is almost automatic with Maple).

3 Large deviation, \( j = n - n^\alpha, \quad \alpha > 1/2 \)

The case \( j = \mathcal{O}(n) \) is analyzed in Timashev [12] and the case \( j = n - c, \ c \) constant, in Grünberg [5].

As previous work for the case \( j = n - n^\alpha \), let us mention Bender [1], Temme [11], Moser and Wyman [10] (see also the comments by Odlyzko in [4], p.1182). They all use, explicitly or not, the Saddle point method. For \( \alpha < 1/2 \), Moser and Wyman (6.9) give an explicit asymptotic expression. For
Figure 2: $J_n(j) \left/ \left[ \frac{1}{\sqrt{2\pi \sigma}} \exp \left[ - \left( \frac{j-M}{\sigma} \right)^2 / 2 \right] \right] \right.,$ color=red, $J_n(j)/R_1,$ color=blue, $n = 3000$

$\alpha > 1/2,$ they first compute in (4.52) the numerical solution $z_n$ of $S'(z_n) = 0$ and give in (4.51) an asymptotic expression. This is rather precise: for $n = 50,$ this gives a precision of order $10^{-4}.$ [1] and [11] also compute numerically $z_n.$ However, all these results do not shed light on the dependence of $[z^j] \phi(z)$ on $n^\alpha.$ This what we want to explicit in this Section. It appears that the range $\alpha > 1/2$ is more delicate than the other range.

Recall that

$$\phi_n(z) = \prod_{0}^{n-1} (z + i) = \frac{\Gamma(z + n)}{\Gamma(z)}.$$

We have

$$G_n(z) := \frac{\Gamma(z + n)}{\Gamma(z) z^{j+1}} = \exp[S(z)],$$

with

$$S(z) = S_1(z) + S_2(z), S_1(z) = \sum_{0}^{n-1} \ln(z + i), S_2(z) = -(j + 1) \ln(z).$$

We first compute $\tilde{z}$ such that

$$S'(\tilde{z}) = 0. \quad (4)$$

We have

$$S'(z) = \psi(z + n) - \psi(z) - \frac{j + 1}{z}.$$

Similarly (we need these expressions later on)

$$S^{(2)}(z) = \psi(1, z + n) - \psi(1, z) + \frac{j + 1}{z^2},$$

$$S^{(k)}(z) = \psi(k - 1, z + n) - \psi(k - 1, z) + (-1)^k(k - 1)! \frac{j + 1}{z^k}.$$
Some experiments with some values for $\alpha$ ($\alpha = 5/8$ is a good choice) show that $\tilde{z}$ must be a combili of $x = n^\alpha$ and $y = n^{1-\alpha}$ and $x \gg y \gg 1$. Note that both $x$ and $y$ are large. The first terms in the asymptotics of $\tilde{z}$ are easy to compute: set $\tilde{z} = n\beta$. Equation (4) leads to

$$\psi(n(1 + \beta)) - \psi(n\beta) = \frac{1}{\beta} - \frac{1}{y\beta} + \frac{1}{n\beta}.$$ 

But $\psi(n) \sim \ln(n)$. So we have

$$\ln \left(1 + \frac{1}{\beta} \right) \sim \frac{1}{\beta} - \frac{1}{y\beta} + \frac{1}{n\beta},$$

or

$$\frac{1}{\beta} - \frac{1}{2\beta^2} \sim \frac{1}{\beta} - \frac{1}{y\beta},$$

or $\beta \sim \frac{y}{2}$.

More generally, we have

$$\beta = \frac{y}{2} \left[ 1 + \frac{a_1}{y} + \frac{a_2}{y^2} + \mathcal{O} \left( \frac{1}{y^3} \right) + 1 \left( 1 + \frac{b_1}{y} + \frac{b_2}{y^2} + \mathcal{O} \left( \frac{1}{y^3} \right) \right) + \frac{1}{x^2} \left( 1 + \frac{c_1}{y} + \frac{c_2}{y^2} + \mathcal{O} \left( \frac{1}{y^3} \right) \right) + \mathcal{O} \left( \frac{1}{x^3} \right) \right].$$

By bootstrapping, we obtain (we give the first terms)

$$\tilde{z} = \frac{ny}{2} \left[ 1 - \frac{4}{3y} + \frac{2}{9y^2} + \frac{8}{135y^3} + \frac{8}{405y^4} + \frac{16}{1701y^5} + \frac{232}{45525y^6} + \frac{32}{3} + \frac{64}{18225y^7} + \mathcal{O} \left( \frac{1}{y^8} \right) \right]$$

$$+ \frac{1}{x} \left[ 1 - \frac{1}{y} + \frac{4}{9y^2} - \frac{16}{135y^3} + \mathcal{O} \left( \frac{1}{y^4} \right) \right]$$

$$+ \frac{1}{x^2} \left[ 1 - \frac{1}{y} + \frac{0}{y^2} + \mathcal{O} \left( \frac{1}{y^3} \right) \right]$$

$$+ \frac{1}{x^3} \left[ 1 + \mathcal{O} \left( \frac{1}{y} \right) \right]$$

$$+ \mathcal{O} \left( \frac{1}{x^4} \right) \right].$$

(5)

Note that the choice of dominant terms in the bracket of (5) depends on $\alpha$. For instance, for $\alpha = 3/4$, the dominant terms (in decreasing order) are

$$\frac{1}{y}, \frac{1}{y^2}, \frac{1}{x}, \frac{1}{x^2}, \frac{1}{xy}, \frac{1}{xy^2}, \frac{1}{xy^3}, \frac{1}{xy^4}, \frac{1}{xy^5}, \frac{1}{xy^6}, \ldots$$

The quality of asymptotic (5) is given in Figure 3 and 4, for $n = 500$, and $x \in [\sqrt{n}, 0.9]$ (first range) so that $y \in [n^{0.1}, \sqrt{n}]$. For some values of $j = n - x$, we show $\tilde{z}/zn$, where, as mentioned, $zn$ is the numerical solution of $S'(zn) = 0$. In the full range $j \in [n - n^{0.9}, n - \sqrt{n}]$, the precision is of order $10^{-5}$, in a restricted range, the precision is of order $10^{-6}$.

Also a comparison of $G_n(\tilde{z})$ and $G_n(zn)$ is given in Figure 5, showing again a precision of order $10^{-6}$.

Now we must compute $S(\tilde{z})$ and its asymptotics. First we compute $\ln(\tilde{z} + i)$, take the asymptotics wrt $x$, sum on $i$, and again take the asymptotics wrt $x$ (recall that $n = xy$). This leads to

$$S_1(\tilde{z}) = x \left[ (-\ln(2) + 2\ln(y) + \ln(x))y - \frac{1}{3} + \frac{4}{405y^2} + \frac{2}{405y^3} + \ldots \right] + y - \frac{2}{3} - \frac{2}{3y} - \frac{49}{135y^2} + \ldots$$

$$+ \frac{1}{x} \left( \frac{y}{2} + \frac{1}{6y} + \ldots \right) + \frac{1}{x^2} \left( \frac{y}{3} + \ldots \right) + \mathcal{O} \left( \frac{y}{x^3} \right).$$
Figure 3: $z_n/\bar{z}, n = 500$, as function of $j$, full range

Figure 4: $z_n/\bar{z}, n = 500$, as function of $j$, restricted range
Figure 5: $G_n(zn)/G_n(\tilde{z}), n = 500$, as function of $j$

Here we provide only a few terms but Maple knows more. Next

$$S_2(\tilde{z}) = x \left[ (\ln(2) - 2\ln(y) - \ln(x))y + \frac{4}{3} - \ln(2) + 2\ln(y) + \ln(x) - \frac{2}{3y} - \frac{94}{405y^2} + \ldots \right]$$

$$- y + \frac{2}{3} + \ln(2) - 2\ln(y) - \ln(x) + \frac{1}{y} + \frac{94}{135y^2} + \ldots$$

$$+ \frac{1}{x} \left( \frac{y}{2} + \frac{1}{6y} + \ldots \right)$$

$$+ \frac{1}{x^2} \left( \frac{y}{3} + \ldots \right)$$

$$+ O \left( \frac{y}{x^3} \right).$$

So, finally

$$S(\tilde{z}) \sim x \left[ 1 - \ln(2) + 2\ln(y) + \ln(x) - \frac{2}{3y} + \ldots \right]$$

$$+ \ln(2) - 2\ln(y) - \ln(x) + \frac{1}{3y} + \frac{1}{3y^2} + \ldots$$

$$+ \frac{1}{x} \left( -\frac{1}{2} + \frac{1}{3y} - \frac{1}{2y^2} + \ldots \right)$$

$$+ \frac{1}{x^2} \left( -\frac{1}{6} + \frac{19}{18y^2} \ldots \right)$$

$$+ O \left( \frac{1}{x^3} \right).$$

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Now we split $S(\tilde{z})$ into two parts:

$$T_1 = x \left[ 1 - \ln(2) + 2 \ln(y) + \ln(x) - \frac{2}{3y} + \ldots \right] + \ln(2) - 2 \ln(y) - \ln(x),$$

$$T_2 = \frac{1}{3y} + \frac{1}{3y^2} + \ldots + \frac{1}{x} \left( -\frac{1}{2} + \frac{1}{3y} - \frac{1}{2y^2} + \ldots \right) + \frac{1}{x^2} \left( -\frac{1}{6} - \frac{17}{18y^2} \ldots \right) + \mathcal{O} \left( \frac{1}{x^3} \right).$$

Note that the dominant term of $T_1$ is given by

$$T_1 \sim (2 - \alpha)n^\alpha \ln(n). \quad (6)$$

We obtain

$$\exp(S(\tilde{z})) = e^{T_1} e^{T_2} = e^{T_1} T_3,$$

with

$$T_3 = e^{T_2} = 1 + \frac{1}{3y} + \frac{7}{18y^2} + \frac{89}{270y^3} + \frac{18263}{3240y^4} + \frac{98009}{3240y^5} + \frac{9517337}{97200y^6} + \frac{491504273}{2041200y^7} + \ldots$$

$$+ \frac{1}{x} \left( -\frac{1}{2} + \frac{1}{6y} - \frac{7}{12y^2} + \frac{2311}{540y^3} + \frac{112469}{6480y^4} + \frac{5137}{144y^5} + \ldots \right) + \frac{1}{x^2} \left( -\frac{1}{24} - \frac{13}{72y} - \frac{557}{932y^2} + \ldots \right)$$

$$+ \mathcal{O} \left( \frac{1}{x^3} \right).$$

Here we have given all terms compatible with the expansion (5). Also, with more precision,

$$T_1 = x \left[ 1 - \ln(2) + 2 \ln(y) + \ln(x) - \frac{2}{3y} - \frac{2}{9y^2} - \frac{44}{405y^3} - \frac{44}{405y^4} + \frac{40}{27y^5} + \frac{179968}{18225y^6} + \frac{4727552}{127575y^7} + \frac{3436796}{32805y^8} + \frac{5492621728}{22143375y^9} + \ldots \right]$$

$$+ \ln(2) - 2 \ln(y) - \ln(x).$$

Now we must consider $S^{(k)}(\tilde{z})$. By direct expansion, this gives the following expressions (again we provide only the first few terms). We must use up to six derivatives to get a sufficient precision (of
order $x^{-2}$ in the Saddle integrals.

$$S^{(2)}(\tilde{z}) = \frac{1}{x} \left[ \frac{4}{y^3} + \frac{16}{3y^5} + \ldots \right]$$

$$+ \frac{1}{x^2} \left[ -\frac{12}{y^4} - \frac{40}{3y^5} + \ldots \right]$$

$$+ \frac{1}{x^3} \left[ \frac{12}{y^4} + \frac{8}{y^5} + \ldots \right]$$

$$+ \frac{1}{x^4} \left[ -\frac{4}{y^4} + \ldots \right]$$

$$+ O\left(\frac{1}{x^5y^4}\right),$$

(7)

Note that, with $z = \tilde{z}e^{i\theta}$, this leads to

$$S^{(2)}(\tilde{z}) \frac{(z - \tilde{z})^2}{2} \sim -\frac{1}{2} n^\alpha \theta^2.$$

(8)

$$S^{(3)}(\tilde{z}) = \frac{1}{x^2} \left[ -\frac{32}{y^5} + \ldots \right]$$

$$+ \frac{1}{x^3} \left[ \frac{128}{y^6} + \ldots \right]$$

$$+ \frac{1}{x^4} \left[ -\frac{192}{y^6} + \ldots \right]$$

$$+ \frac{1}{x^5} \left[ \frac{128}{y^6} + \ldots \right]$$

$$+ O\left(\frac{1}{x^6y^5}\right),$$

$$S^{(4)}(\tilde{z}) = \frac{1}{x^3} \left[ \frac{288}{y^8} + \ldots \right]$$

$$+ \frac{1}{x^4} \left[ -\frac{1440}{y^8} + \ldots \right]$$

$$+ \frac{1}{x^5} \left[ \frac{2880}{y^8} + \ldots \right]$$

$$+ \frac{1}{x^6} \left[ -\frac{2880}{y^8} + \ldots \right]$$

$$+ O\left(\frac{1}{x^7y^6}\right),$$

$$S^{(5)}(\tilde{z}) = \frac{1}{x^4} \left[ -\frac{3072}{y^{10}} + \ldots \right]$$

$$+ \frac{1}{x^5} \left[ \frac{18432}{y^{10}} + \ldots \right]$$

$$+ O\left(\frac{1}{x^6y^{10}}\right),$$

$$S^{(6)}(\tilde{z}) = \frac{1}{x^5} \left[ \frac{38400}{y^{12}} + \ldots \right]$$

$$+ \frac{1}{x^6} \left[ \frac{268800}{y^{12}} + \ldots \right]$$

$$+ O\left(\frac{1}{x^7y^{12}}\right).$$
To check the quality of asymptotic (5), we give in Figure 6 the comparison between the expression (7) and $S^{(2)}(\tilde{z})$. The precision is of of order $10^{-2}$.

![Graph showing the quotient of the expression (7) and $S^{(2)}(\tilde{z})$ as a function of $j$, $n = 500$.]

Figure 6: The quotient of the expression (7) and $S^{(2)}(\tilde{z})$ as function of $j$, $n = 500$

In a restricted range, given in Figure 7, the precision is of order $10^{-4}$. \( \alpha \leq 0.84 \) in this range.

We proceed now as in Section 2. Again, the justification of the integration procedure is given in the Appendix. This leads to

$$
\tau = \frac{y^2 \sqrt{x}}{2} \left[ u a_1 + \frac{u^2 a_2}{x^{1/2}} + \frac{u^3 a_3}{x} + \frac{u^4 a_4}{x^{3/2}} + \frac{u^5 a_5}{x^2} + O \left( \frac{u^6}{x^{5/2}} \right) \right].
$$

We give only $a_1$:

$$
a_1 = 1 - \frac{2}{3y} - \frac{2}{9y^2} + \ldots + \frac{1}{x} \left( \frac{3}{2} - \frac{4}{3y} + \ldots \right) + \frac{1}{x^2} \left( \frac{15}{8} - \frac{7}{4y} + \ldots \right) + O \left( \frac{1}{x^3} \right).
$$

This leads to

$$
\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-u^2/2} \tau'(u) du = \frac{y^2 \sqrt{x}}{2} T_4,
$$

with

$$
T_4 = 1 - \frac{2}{3y} - \frac{2}{9y^2} + \ldots + \frac{1}{x} \left( \frac{5}{12} - \frac{11}{18y} + \ldots \right) + \frac{1}{x^2} \left( \frac{73}{288} - \frac{133}{432y} + \ldots \right) + \frac{1}{x^3} \left( \frac{721}{576} + \ldots \right) + O \left( \frac{1}{x^4} \right).
$$

Set

$$
T_5 := \frac{1}{\sqrt{2\pi}} \frac{y^2 \sqrt{x}}{2} e^{T_1}.
$$

This leads to

$$
[z^j] \phi_n(z) \sim T_5 T_3 T_4. \quad \tag{9}
$$

We can of course combine $T_3$ and $T_4$:

$$
T_6 := T_3 T_4 = 1 - \frac{3}{3y} - \frac{1}{18y^2} - \frac{1}{30y^3} + \frac{17207}{3240y^4} + \ldots + \frac{1}{x} \left( -\frac{1}{12} + \frac{1}{36y} - \frac{35}{216y^2} + \frac{15029}{3240y^3} + \ldots \right)
$$

$$
+ \frac{1}{x^2} \left( \frac{1}{288} - \frac{1}{864y} + \frac{3527}{5184y^2} + \ldots \right) + O \left( \frac{1}{x^3} \right).
$$
Figure 7: The quotient of the expression (7) and $S^{(2)}(\tilde{z})$, as function of $j$, $n = 500$. Restricted range, $\alpha \leq .84$

We have made several experiments with (9), with $n$ up to 500. The result is unsatisfactory, only values of $x$ of order $\sqrt{n}$ give reasonable results. Also using $e^{T_2}$ instead of $T_3$ does not improve the precision. Actually, only very large values of $n$ lead to good precision. So we turn to another formulation: instead of using $e^{T_1}T_3$ for $e^{S(\tilde{z})}$, we plug directly $\tilde{z}$ into $G_n(\tilde{z})$, i.e. we set

$$T_7 = G_n(\tilde{z}),$$

leading to

$$[z^j]\phi_n(z) \sim \frac{1}{\sqrt{2\pi}} \frac{y^2\sqrt{x}}{2} T_7 T_4 =: T_8 \text{ say}.$$

For $n = 500$, using two and three terms in $T_4$, we give in Figures 8 and 9, the quotient $[z^j]\phi_n(z)/T_8$. The precision is of order $10^{-5}$.

4 Appendix. Justification of the integration procedure

4.1 The central region

We proceed as in Flajolet and Sedgewick [3, ch. VIII]. We can choose here $\tilde{z} = 1$. This leads, with $z = e^{i\theta}$, to

$$S(z) \sim S_0(z) + O\left(\sqrt{\ln(n)\theta}\right) + \text{constant term},$$
Figure 8: The quotient $[z^j] \phi_n(z)/T_8$, two terms in $T_4$, as function of $j$, $n = 500$

Figure 9: The quotient $[z^j] \phi_n(z)/T_8$, three terms in $T_4$, as function of $j$, $n = 500$
with

\[ S_0(z) = \sum_{k=1}^{n-1} \ln[e^{i\theta} + k] - H_n i\theta \]

\[ \sim \sum_{i=k}^{n-1} \frac{1}{1 + k} e^{i\theta} - 1 - \frac{1}{2} \sum_{k=1}^{n-1} \left[ \frac{1}{1 + k} e^{i\theta} - 1 \right]^2 - H_n i\theta + O(\theta^3) \]

\[ \sim H_n [e^{i\theta} - 1 - i\theta] + O(\theta^2). \]

Set

\[ h(\theta) := e^{i\theta} - 1 - i\theta. \]

We have

\[ h(\theta) \sim -\frac{\theta^2}{2}, \]

which conforms to (3).

The function \( h(\theta) \) is the same as in [3, Ex.VIII.3], which proves the validity of our integration procedure: we use here \( H_n \sim \ln(n) \) instead of \( n \). The complete asymptotic expansion is justified as in [3, Ex.VIII.4].

4.2 The non-central region

We choose here \( \tilde{z} = \frac{ny}{2} = \frac{n^2 - \alpha}{2} := \delta \), say. We have

\[ \frac{1}{2} < \alpha < 1, \]

\[ n^\alpha = \frac{n^2}{2\delta}, \]

\[ n^2 \gg \delta \gg n \gg n^\alpha. \]

Set \( z = \delta e^{i\theta} \), this leads, with Euler-Maclaurin formula, with the first correction (the other corrections are negligible), to

\[ S(z) \sim \sum_{k=1}^{n-1} \ln \left[ \delta e^{i\theta} + k \right] - (n - n^\alpha) i\theta - (n - n^\alpha) \ln(\delta) - \frac{1}{2} \ln \left( n + \delta e^{i\theta} \right) + \frac{1}{2} \ln \left( \delta e^{i\theta} \right) \]

\[ \sim \ln \left[ n + \delta e^{i\theta} \right] \left[ n + \delta e^{i\theta} \right] - n - \delta e^{i\theta} \ln \left[ \delta e^{i\theta} \right] - \left[ n - \frac{n^2}{2\delta} \right] (i\theta + \ln(\delta)) - \frac{1}{2} \ln \left[ n + \delta e^{i\theta} \right] + \frac{1}{2} \ln \left[ \delta e^{i\theta} \right]. \]

Set now \( n = \rho \delta, \rho = 2n^{\alpha-1} \ll 1 \) and expand wrt \( \rho \). This gives

\[ S(z) \sim \rho \left[ -\frac{1}{2} e^{-i\theta} \right] \]

\[ + \rho^2 \left[ \frac{1 + i\theta e^{i\theta}}{2e^{i\theta}} + \frac{1}{4} e^{-2i\theta} + \frac{1}{2} \delta \ln(\delta) \right] \]

\[ + \rho^3 \left[ -\frac{\delta}{6} e^{-2i\theta} - \frac{1}{6} e^{-3i\theta} \right] \]

\[ + O(\delta \rho^4). \]

Note that the dominant constant contribution is given by \( \frac{1}{2} \rho^2 \delta \ln(\delta) = (2 - \alpha) n^\alpha \ln(n), \) which conforms to (6). The first term gives a variable part \( O(\rho) \). The second term gives a variable part \( 2n^\alpha h(\theta) + O(\rho^2) \), with

\[ h(\theta) := \frac{1 + i\theta e^{i\theta}}{2e^{i\theta}}. \]
The third term gives \( O(n^{2\alpha-1}) \ll n^\alpha \). Note that \( 2n^\alpha h(\theta) \sim -\frac{1}{2}n^\alpha \theta^2 \), which conforms to (8). The function \( |e^{h(\theta)}| = e^{\cos(\theta)/2} \) is unimodal with peak at 0 and \( h(0) = 1/2 \). Let us introduce a splitting value \( \theta_0 \) such that \( n^\alpha \theta_0^3 \to 0 \), \( n^\alpha \theta_3 \to 0 \), \( n \to \infty \). For instance, we choose \( \theta_0 = n^{\beta}, \beta = -\frac{5\alpha}{12} \). By unimodality property of the cosine, the tail integral

\[
K_n^{(1)} := \int_{\theta_0}^{2\pi - \theta_0} e^{2n^\alpha (h(\theta) - 1/2)} d\theta
\]

is such that

\[
|K_n^{(1)}| = O\left(e^{n^\alpha [\cos(\theta_0) - 1]}\right) = O\left(e^{-Cn^\alpha/6}\right)
\]

for some \( C > 0 \). The tail integral is exponentially small.

As \( h(\theta) \sim -\frac{\theta^2}{4} \), the central approximation and the tail completion are immediate.

5 Conclusion

Using an almost mechanized program in Maple, we have obtained some asymptotic expressions for Stirling numbers in central and non-central regions. We intend to use these techniques in other non-central ranges.

References


