A refined and asymptotic analysis of optimal stopping problems of Bruss and Weber

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Abstract

The classical secretary problem has been generalized over the years into several directions. In this paper we confine our interest to those generalizations which have to do with the more general problem of stopping on a last observation of a specific kind. We follow Dendievel [10], [11], (where a bibliography can be found) who studies several types of such problems, mainly initiated by Bruss [3] and [5], Weber [17] and [18]. Whether in discrete time or continuous time, whether all parameters are known or must be sequentially estimated, we shall call such problems simply *Bruss-Weber problems.* Our contribution in the present paper is a refined analysis of several problems in this class and a study of the asymptotic behaviour of solutions.

The problems we consider center around the following model. Let $X_1, X_2, \dots, X_n$ be a sequence of independent random variables which can take three values: $\{+1, -1, 0\}$. Let $p := \mathbb{P}(X_i = 1), p' := \mathbb{P}(X_i = -1), \tilde{q} := \mathbb{P}(X_i = 0), p \geq p'$, where $p + p' + \tilde{q} = 1$. The goal is to maximize the probability of stopping on a value $+1$ or -1 appearing for the last time in the sequence. Following a suggestion by Bruss, we have also analyzed an x-strategy with incomplete information: the cases p known, n unknown, then n known, p unknown and finally n, p unknown are considered. We also present simulations of the corresponding complete selection algorithm.

Keywords: Stopping times, Unified Approach to best choice, Odds-algorithm, Optimal solutions, x-Strategy, Asymptotic expansions, Incomplete information.

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1 Introduction

The classical secretary problem has been generalized over the years into several directions. In this paper we confine our interest to those generalizations which have to do with the more general problem of stopping on a last observation of a specific kind. We follow Dendievel [10], [11], (where a bibliography can be found) who studies several types of such problems, mainly initiated by Bruss [3], [5] and Weber [17], [18]. Whether in discrete time or continuous time, whether all parameters are known or must be sequentially estimated, we shall call such problems simply *Bruss-Weber problems.*

Bruss [5] studied the case of stopping on a last 1 in a sequence of n independent random variables $X_1, X_2, \dots, X_n$, taking values $\{1, 0\}$. This led to the versatile odds-algorithm and also to a similar method in continuous-time, allowing for interesting applications in different domains, as e.g. in investment problems studied in Bruss and Ferguson [7]. See also Szajowski and Łebek [15]. Moreover, Bruss and Louchard [8] studied the case where the odds are unknown and have to be sequentially estimated, showing a convincing stability for applications.

Weber (R.R. Weber, University of Cambridge), considered the model of iid random variables taking values in $\{+1, -1, 0\}$. The goal is to maximize the probability of stopping on a value $+1$ or -1 appearing for the last time in the sequence. The background was as follows.

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When teaching the odds-algorithm in his course (see section 6 of his course on optimization and control [17]), Weber proposed the following problem to his students:

A financial advisor can impress his clients if immediately following a week in which the FTSE index moves by more than 5% in some direction he correctly predicts that this is the last week during the calendar year that it moves more than 5% in that direction

Suppose that in each week the change in the index is independently up by at least 5%, down by at least 5% or neither of these, with probabilities p, p and $1 - 2p$ respectively ($p \leq 1/2$). He makes at most one prediction this year. With what strategy does he maximize the probability of impressing his clients?

The solution of this interesting problem is easy but can only be partially retrieved from the odds-algorithm.

Weber [18] then discussed with Bruss several more difficult versions of this problem, some of them studied in Dendievel's PhD thesis [11].

Let us also mention shortly related work: Hsiau and Yang [12] have studied the problem of stopping on a last 1 in a sequence of Bernoulli trials in a Markovian framework, where the value taken by the kth variable is influenced by the value of the the $(k-1)$th variable. Ano and Ando [1], generalizing the model of Bruss [4], consider options arising according to a Poisson process with unknown intensity but only available with a fixed probability p. Tamaki [16] generalized the odds-algorithm by introducing multiplicative odds in order to solve the problem of optimal stopping on any of a fixed number of last successes. Surprising coincidences of lower bounds for odds-problems with multiple stopping have been discovered by Matsui and Ano [14], generalizing Bruss [6]. A more specific interesting problem of multiple stopping in Bernoulli trials with a random number of observations was studied by Kurushima and Ano [13].

Let $p := \mathbb{P}(X_i = 1), p' := \mathbb{P}(X_i = -1), \tilde{q} := \mathbb{P}(X_i = 0), p \geq p'$, where $p + p' + \tilde{q} = 1$.

A first problem studied in [10] is to maximize for a fixed number n of variables the success probability $w_{j,k}, j \geq k$ with the following strategy: we observe $X_1, X_2, \ldots$. Wait until $i = k$. From k on, if $X_i = -1$ we select X_i and stop. If not we proceed to the next random variable and start the algorithm again. If no -1 value was found before j, then, from j on, if $X_i = +1$ or $X_i = -1$ we select this variable and stop. If none was found (all $X_i = 0$ from j to n) then we fail. The goal is to find j^*, k^* such that w_{j^*,k^*} is maximum. In [10], explicit expressions for $w_{j,k}, w_{j,j}$ are given and j^*, k^* are numerically computed for given n. Dendievel also proves that the problem is monotone in the sense of Assaf and Samuel-Cahn [2]: if at a certain time it is optimal to stop on a 1 (respectively on a -1), then it is optimal to stop on a 1(respectively on a -1) at any later time index. Also, it is proved in [10], that if $p \geq p'$ then $j^* \geq k$.

Our contribution is the following: in Section 2, we provide explicit optimal solutions in a continuous model and in the present discrete case for $p > p'$ and $p = p'$.

Another problem, initiated by a model of Bruss in continuous time, and leading to the 1/e-law of best choice (Bruss [3]) is a problem in continuous time, now with a fixed total number of variables n with possible values in $0, -1, 1$. More precisely, let $U_i, i = 1, 2, \ldots, n$ be independent random variables uniformly distributed on the interval $[0, 1]$. Let $T_i = U_{\{i\}}$: T_i is the ith order statistic of the U_i's. T_i is the arrival time of X_i. The strategy is to wait until some time x_n^* and from x_n^* on, we select the first $X_i = +1$ or $X_i = -1$, using the previous algorithm with $p = p'$. Following Bruss [5], we call this strategy an x-strategy. In [10], for this problem, the author gives the optimal x_n^* and the corresponding success probability P_n^*.

In Section 3 we provide some asymptotic expansions for this x-strategy's parameters, for $p = p'$. We also consider the success probability for small p and for the case $p > p'$.

In Section 4, following a suggestion by Bruss, we have analyzed an x-strategy with incomplete information: the cases p known, n unknown, then n known, p unknown and finally n, p unknown are considered. We also present simulations of the complete selection algorithm.

2 The optimal solution

In this Section, we analyze explicitly the optimal solutions in the continuous and discrete case for $p > p'$ and $p = p'$. The following notations will be used in the sequel: $q := 1 - p, q' = 1 - p', \tilde{q} = 1 - p - p'$.

2.1 The optimal solution, continuous case, $p > p'$

Let us first consider $p > p', j \geq k$. The success probabilities satisfy the following forward recurrence equations (these are easily obtained from the stopping times characterizations):

$$w_{j,j} = pq^{n-j} + p'q'^{n-j} + \tilde{q}w_{j+1,j+1}, \quad w_{n,n} = p + p', \tag{1}$$

$$w_{j,k} = p'q'^{n-k} + q'w_{j,k+1}. \tag{2}$$

The solutions, already given in Dendievel [10], are

$$w_{jj} = (p^2\, q^{n-j+1} - p^2\, \tilde{q}^{n-j+1} + p'^2\, q'^{n-j+1} - p'^2\, \tilde{q}^{n-j+1})/(p'\, p), \tag{3}$$

$$w_{j,k} = (j-k)\, p'\, q'^{n-k} + q'^{j-k}\left(\frac{p\,(q^{n-j+1} - \tilde{q}^{n-j+1})}{p'} + \frac{p'\,(q'^{n-j+1} - \tilde{q}^{n-j+1})}{p}\right). \tag{4}$$

If $j \leq k$, we use

$$\begin{aligned} w_{k,j} &:= (k-j)\, p\, q^{n-j} \\ &+ q^{k-j}\left(\frac{p'\,(q'^{n-k+1} - \tilde{q}^{n-k+1})}{p} + \frac{p\,(q^{n-k+1} - \tilde{q}^{n-k+1})}{p'}\right). \end{aligned}$$

Simplification using generating functions

We shall show that these expressions can be nicely derived by using backward generating functions. Let $F(z) := \sum_{j=-\infty}^{n-1} z^{n-j} w_{j,j}$. From (1), we have

$$F(z) - p - p' - \frac{p'\, q'\, z}{1 - z + p'\, z} - \frac{p\, q\, z}{1 - z + z\, p} - \tilde{q}\, z\, F(z) = 0,$$

the solution of which is

$$\begin{aligned} F(z) &= \frac{-p'\, z + p + p' + 2\, p\, p'\, z - z\, p}{(1 - z + z\, p)\,(1 - z + p'\, z)\,(1 - z + z\, p + p'\, z)} \\ &= -\frac{(p^2 + p'^2)\, \tilde{q}}{p'\, p\,(1 - z + z\, p + p'\, z)} + \frac{p'\, q'}{p\,(1 - z + p'\, z)} + \frac{p\, q}{p'\,(1 - z + z\, p)}. \end{aligned}$$

This immediately leads to (3). Similarly, let $F_j(z) := \sum_{k=-\infty}^{j-1} z^{j-k} w_{j,k}$. From (2) this satisfies

$$\begin{aligned} &F_j(z) - \big(p^2\, q^{n-j+1} - p^2\, \tilde{q}^{n-j+1} + p'^2\, q'^{n-j+1} \\ &- p'^2\, \tilde{q}^{n-j+1}\big)\,/(p'\, p) - \frac{p'\, z}{q'^{-n+j-1}\,(1 - z + p'\, z)} - q'\, z\, F_j(z) = 0, \end{aligned}$$

the solution of which, expanded into partial fractions, leads to

$$\begin{aligned} F_j(z) &= \big(-p'^3\, q'^{n-j} + p'^3\, \tilde{q}^{n-j} - p\, p'^2\, q'^{n-j} + p'^2\, q'^{n-j} + p\, p'^2\, \tilde{q}^{n-j} - p'^2\, \tilde{q}^{n-j} \\ &+ p'\, p^2\, \tilde{q}^{n-j} - p^2\, \tilde{q}^{n-j} + p^2\, q^{n-j} - p^3\, q^{n-j} + p^3\, \tilde{q}^{n-j}\big)\,/((1 - z + p'\, z)\, p'\, p) + \frac{p'\, q'^{n-j}}{(1 - z + p'\, z)^2}. \end{aligned}$$

This simplifies as

$$F_j(z) = \big(p^2 q q^{n-j} + p'^2 \tilde{q} q'^{n-j} - (p^2 + p'^2)\tilde{q}^{n-j+1}\big)\,/((1 - z + p'z)pp') + \frac{p'\, q'^{n-j}}{(1 - z + p'\, z)^2}.$$

Now from (2) the presumed generating function is given by

$$\begin{aligned} Fth_j(z) &= \frac{p'\,z}{q'^{(-n+j-1)}\,(-1+q'\,z)^2} \\ &\quad - q'\left(\frac{p}{p'}\left(\frac{1}{q^{(-n+j-1)}} - \frac{1}{\tilde{q}^{(-n+j-1)}}\right) + \frac{p'}{p}\left(\frac{1}{q'^{(-n+j-1)}} - \frac{1}{\tilde{q}^{(-n+j-1)}}\right)\right)\frac{z}{(-1+q'\,z)}. \end{aligned}$$

Identification with $F_j(z)$ is immediate.

Computation of the optimal values j^*, k^*

Let us now turn to the main object of this Section which is the computation of the optimal values j^*, k^*. It is proved in [10] that, if $p > p'$ then $j^* \geq j^*$. Actually, setting $j = n-C, k = n-D$ in (3),(4), we see that $w_{j,k}, w_{k,j}$ *do not depend on* n . We have, with $C \leq D$, and using C, D as continuous variables,

$$w_{C,D} := (-C+D)\,p'\,q'^D + q'^{-C+D}\left(\frac{p\,(q^{C+1} - \tilde{q}^{C+1})'}{p} + \frac{p'\,(q'^{C+1} - \tilde{q}^{C+1})}{p}\right),$$

and if $D \leq C$,

$$w_{D,C} := (-D+C)\,p\,q^C + q^{-D+C}\left(\frac{p'\,(q'^{D+1} - \tilde{q}^{D+1})}{p} + \frac{p\,(q^{D+1} - \tilde{q}^{D+1})}{p'}\right).$$

The optimal value C^* is the (unique) solution of

$$\phi_1(C^*) = 0, \tag{5}$$

$$\phi_1(C) := \frac{\partial w_{C,D}}{\partial C} q'^{C-D} pp' = -\tilde{q}\,(p^2 + p'^2)\,(-\ln(q') + \ln(\tilde{q}))\tilde{q}^C + p^2\,q\,(-\ln(q') + \ln(q))q^C - p'^2 pq'^C. \tag{6}$$

First of all, we have $\tilde{q} < q < q'$, $p' < p$ for $0 \leq p \leq 1/2$, $p' < 1-p$ for $1/2 \leq p \leq 1$. Dividing Eq. (6) by q'^C, we see that $\phi_1(C) \sim \phi_{as}(C) = -p'^2 pq'^C, C \to \infty$ which is negative. A plot of $\phi_1(C)$, for $p = 0.09, p' = 0.05$ is given in Figure 1, together with $\phi_{as}(C)$, showing numerically a unique maximum, but we need a formal proof.

We would like to have $\phi_1(0) > 0$, this would imply the existence of C^*. A plot of $\phi_1(0)$ (satisfying the constraints on (p, p')) is given in Figure 2. We see that there exists a curve $p' = \gamma_1(p)$, given in Figure 3, such that $\phi_1(0) < 0$ if $p' > \gamma_1(p)$. In this case, we must choose $C^* = 0$. Otherwise, we know that C^* does exist. The extremal points of $\gamma_1(p)$ are $(0.4170224307\ldots, 0.4170224307\ldots), (0.63212005588\ldots, 0)$.

Finally, we must prove the uniqueness of C^*. By dividing Eq.(5) by q'^C, we obtain, with $\tilde{r} := \tilde{q}/q', r := q/q', \tilde{r} < r$,

$$A_1 \tilde{r}^C = A_2 r^C + A_3,$$

where A_1, A_2, A_3 do not depend on C. On both sides, we have strictly convex/concave functions of C which ensure the uniqueness of C^*.

Interestingly, C^* does not depend on D. The optimal value D^* is the solution, for $C = C^*$, of

$$\begin{aligned} \frac{\partial w_{C,D}}{\partial D} q'^{-D} pp' &= p'^2\,p - p'^2 \ln(q')\,C\,p + p'^2 \ln(q')\,D\,p + \ln(q')\,q'^{-C}\,p^2\,q^{C+1} \\ &\quad - \ln(q')\,q'^{-C}\,p^2\,\tilde{q}^{C+1} + p'^2\,q' \ln(q') - \ln(q')\,q'^{-C}\,p'^2\,\tilde{q}^{C+1} = 0, \end{aligned}$$

this gives

$$D = \phi_2(C) :== \left(-\frac{p\,q\,q^C}{p'^2} + \frac{\tilde{q}\,(p^2 + p'^2)}{p'^2\,p}\tilde{q}^C\right) q'^{-C} + \frac{-p + \ln(q')\,C\,p - q' \ln(q')}{\ln(q')\,p}, \tag{7}$$

and $D^* = \phi_2(C^*)$.

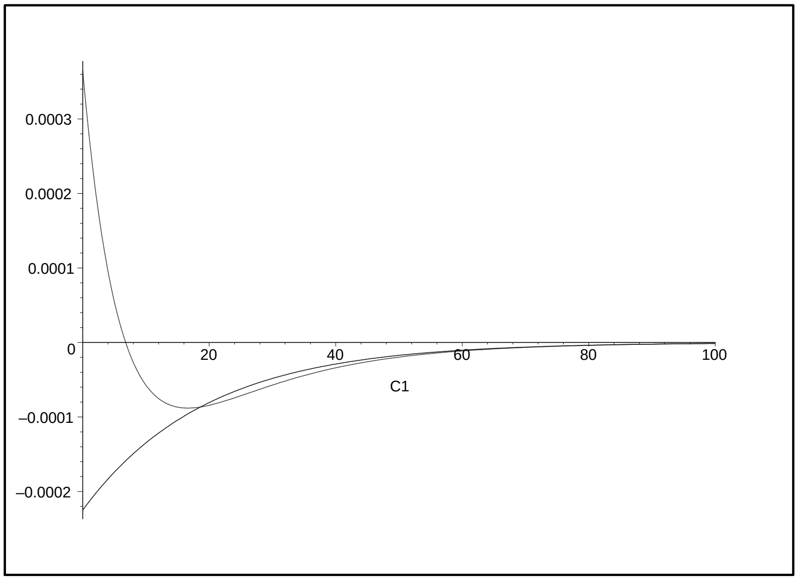

Figure 1: $\phi_1(C)$, $p = 0.09, p' = 0.05$, together with $\phi_{as}(C)$ (lower curve)

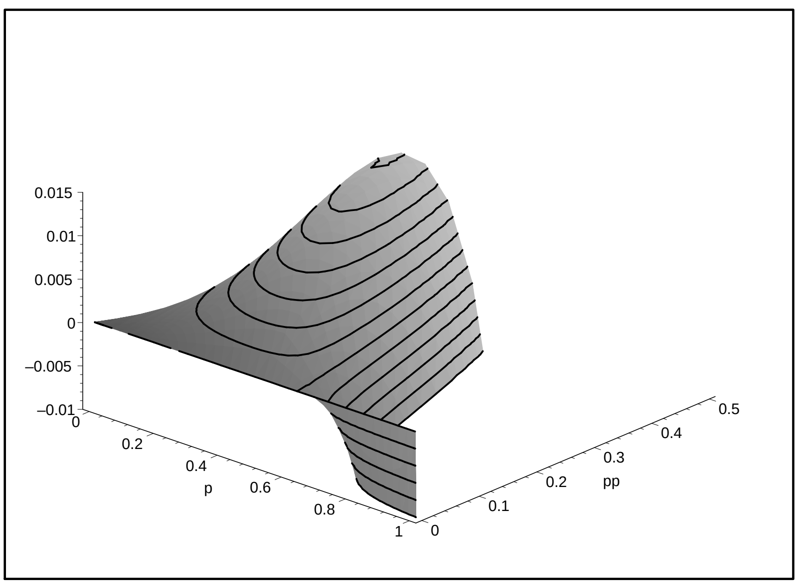

Figure 2: A plot $\phi_1(0)$ defined in Eq.6 as a function of p, p'

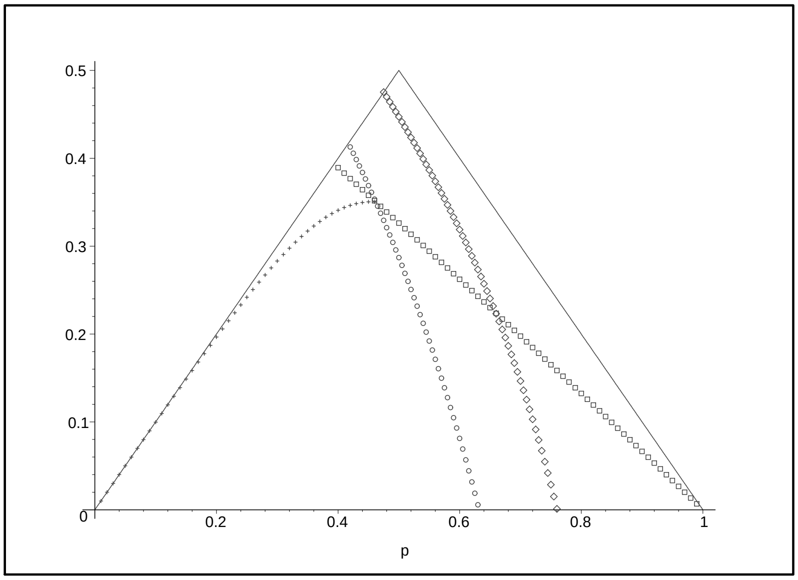

Figure 3: The graphic shows the functions $\gamma_1(p)$ (circles), $\gamma_2(p)$ (box), $\gamma_3(p)$ (cross), $\gamma_4(p)$ (diamonds) defined in the text with the constraints on (p, p')

The acceptance regions

1. Curiously enough, even if we must choose $C^* = 0$ (see above), D^* *is not necessarily non-negative!* If we solve $\phi_2(0) = 0$ w.r.t p' for each p, we obtain a second curve $p' = \gamma_2(p)$ also given in Figure 3. The extremal points of $\gamma_2(p)$ are $(0.3934693403\ldots, 0.3934693403\ldots), (1, 0)$. If $p' > \gamma_2(p)$, then we must choose $D^* = 0$ which means waiting until X_n. Notice that the two curves do cross.

2. Even more interesting, even if $C^* > 0$, D^* *is not necessarily* $> C^*$. If we solve $\{\phi_1(C^*) = 0, \phi_2(C^*) = C^*\}$ w.r.t. $\{C^*, p'\}$, we obtain a third curve $p' = \gamma_3(p)$ also given in Figure 3. If $p' > \gamma_3(p)$, we must choose the optimal point on the diagonal: see the remark below at the end of Section 2.3. The intersection of $\gamma_1, \gamma_2, \gamma_3$ is given by $p_\bullet = 0.461926509410\ldots, p'_\bullet = 0.350346565861\ldots$.

3. Finally, if we stay above the curve $\gamma_2(p)$, we obtain $C^* < 0$. For instance, for $C^* = -0.3$, if we solve $\phi_1(-0.3) = 0$ w.r.t p' for each p, we obtain a fourth curve $p' = \gamma_4(p)$ also given in Figure 3. The extremal points of $\gamma_4(p)$ are $(0.4751561101\ldots, 0.4751561101\ldots), (0.7603489635, 0)$. $\gamma_4(p)$ is of course not practically useful in our analysis (we must have $C^* \geq 0$), but it has some interesting asymptotic properties that we detail in Appendix 6.

A useful table summarizing acceptance regions

The following table 1 shows the different $\{p, p'\}$ regions and their corresponding C^*, D^* characteristics.

p, p'	Theoretical C^*, D^*	Practical C^*, D^*
$p' > \gamma_1(p), p' > \gamma_2(p)$	$C^* < 0, \phi_2(0) < 0$	$C^* = 0, D^* = 0$
$p' = \gamma_2(p), p > p_\bullet$	$C^* < 0, \phi_2(0) = 0$	$C^* = 0, D^* = 0$
$p' > \gamma_1(p), p' < \gamma_2(p), p > p_\bullet$	$C^* < 0, \phi_2(0) > 0$	$C^* = 0, D^* = \phi_2(0)$
$p' = \gamma_1(p), p < p_\bullet$	$C^* = 0, \phi_2(0) < 0$	$C^* = 0, D^* = 0$
$p = p_\bullet, p' = p'_\bullet$	$C^* = 0, \phi_2(0) = 0$	$C^* = 0, D^* = 0$
$p' = \gamma_1(p), p > p_\bullet$	$C^* = 0, \phi_2(0) > 0$	$C^* = 0, D^* = \phi_2(0)$
$p' > \gamma_2(p), p' < \gamma_1(p), p < p_\bullet$	$C^* > 0, \phi_2(0) < 0$	$C^*, D^* = C^*$
$p' = \gamma_2(p), p' > \gamma_3(p), p < p_\bullet$	$C^* > 0, \phi_2(0) = 0$	$C^*, D^* = C^*$
$p' < \gamma_2(p), p' > \gamma_3(p), p < p_\bullet$	$C^* > 0, \phi_2(0) > 0, \phi_2(C^*) < C^*$	$C^*, D^* = C^*$ [1]
$p' < \gamma_1(p), p' < \gamma_2(p), p' < \gamma_3(p)$	$C^* > 0, \phi_2(C^*) > C^*$	$C^*, D^* = \phi_2(C^*)$

Table 1: $\{p, p'\}$ regions and their corresponding C^*, D^* characteristics

As an illustration of the last line of Table 1, a plot of $w_{C,D}, p = 0.09, p' = 0.05, C \le D$ is given in Figure 4 as well as $w_{D,C}, C \ge D$. Also $w_{C^*,D^*} = 0.529979034749\ldots, p = 0.09, p' = 0.05$.

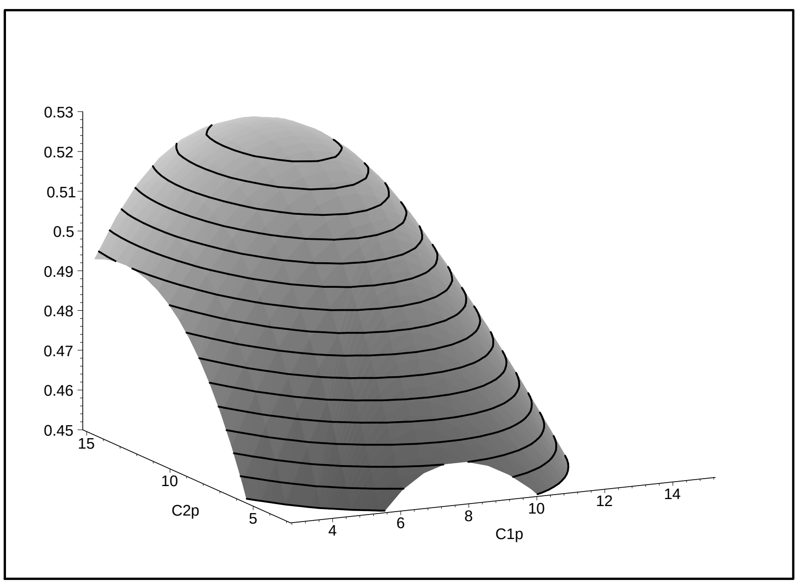

Figure 4: $w_{C,D}, C \le D$, $w_{D,C}, C \ge D$, $p = 0.09, p' = 0.05$

[1]see the remark below at the end of Section 2.3

2.2 The optimal solution in the discrete case for $p > p'$

We must now investigate the discrete values, close to C^*, D^*, leading to the optimal success probabilities. Of course, it is not the discrete values just closest to C^*, D^* . We must compute the corresponding numerical values of $w_{C,D}$. For instance, with $p = 0.09, p' = 0.05$,we have $C^* = 6.785137352\ldots, D^* = 11.88032106\ldots$. The Figure 5 shows $C^*, \phi_2(C)$ and some closest discrete points. It appears that, numerically, the discrete solution is $C_d^* = 7, D_d^* = 12$. This fits with the numerical experiments done in [10], with $w_{j,k}, n = 40$. This gives $w_{C_d^*,D_d^*} = 0.529870739109\ldots$, not far from the continuous value w_{C^*,D^*}.

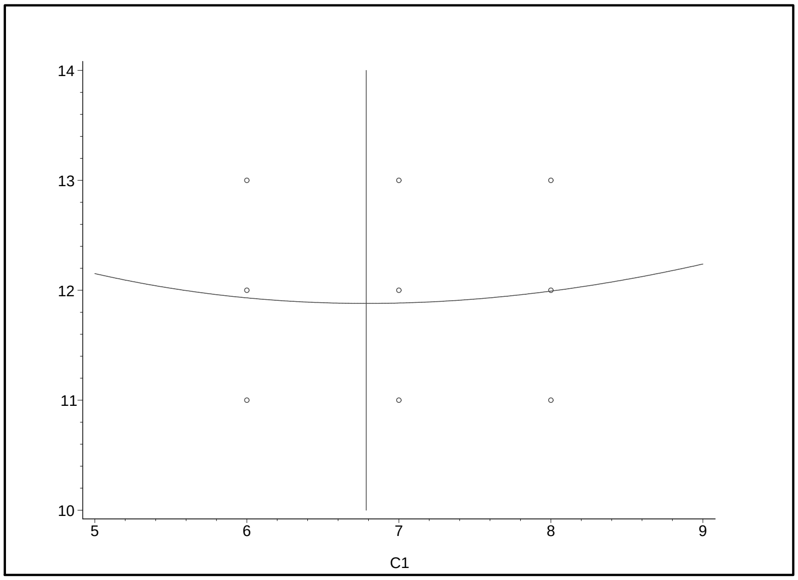

Figure 5: C^* (vertical line), $\phi_2(C)$ (curved line) , $p = 0.09, p' = 0.05$, and some closest discrete points

Notice that two discrete couples can lead to the same optimal solution. For instance, with $p' = 0.05$,$w_{6,12} - w_{7,12}$ is null for $p = 0.09396249862111\ldots$.

2.3 The optimal solution for $p = p'$

Notice that, if $p = p'$, the coefficient of q^C in (6) is null and the coefficient of $\tilde{q}^C$ becomes $T := 2\tilde{q}p'^2(\ln(q) - \ln(\tilde{q}))$. Hence we have the explicit solution

$$C^*_{eq} = \frac{\ln\left(\dfrac{(1-2\,p)\,(2\ln(1-p)\,p^2 - 2\,p^2\ln(1-2\,p))}{p^3}\right)}{\ln\left(\dfrac{1-p}{1-2\,p}\right)}. \tag{8}$$

From (7), we obtain

$$\phi_{2,eq}(C) = \left(-p + p\ln(q)\,C - 2\ln(q) + 2\ln(q)\,p + 2\ln(q)\,q^{-C}\,(1-2\,p)^{C+1}\right)/(p\ln(q)),$$

and again, $D^*_{eq} = \phi_{2,eq}(C^*_{eq})$. $w_{C,D}, w_{C,C}$ become now

$$\begin{aligned} w_{eq,C,D} &= (D-C)pq^D + q^{D-C}2\left(q^{D+1} - \tilde{q}^{C+1}\right), \\ w_{eq,C,C} &= 2\left(q^{C+1} - \tilde{q}^{C+1}\right). \end{aligned} \tag{9}$$

Of course, we must use $w_{eq,C,C}$ in our case, and the solution of $\frac{\partial w_{eq,C,C}}{\partial C} = 0$ is given by

$$C^*_{diag} = -(\ln(\ln(q)/\ln(\tilde{q})) + \ln(q) - \ln(\tilde{q}))/(\ln(q) - \ln(\tilde{q})).$$

Figure 6 shows, for $p = p' = 0.09, C^*_{eq} = 6.15156149309\ldots, \phi_{2,eq}(C), D^*_{eq} = 6.13502664794\ldots, C^*_{diag} = 6.14370678209\ldots$ the point $(6,6)$ and the diagonal. Notice that the point (C^*_{eq}, D^*_{eq}) is *below the diagonal.* Of course, only the part $C \leq D$ is relevant.

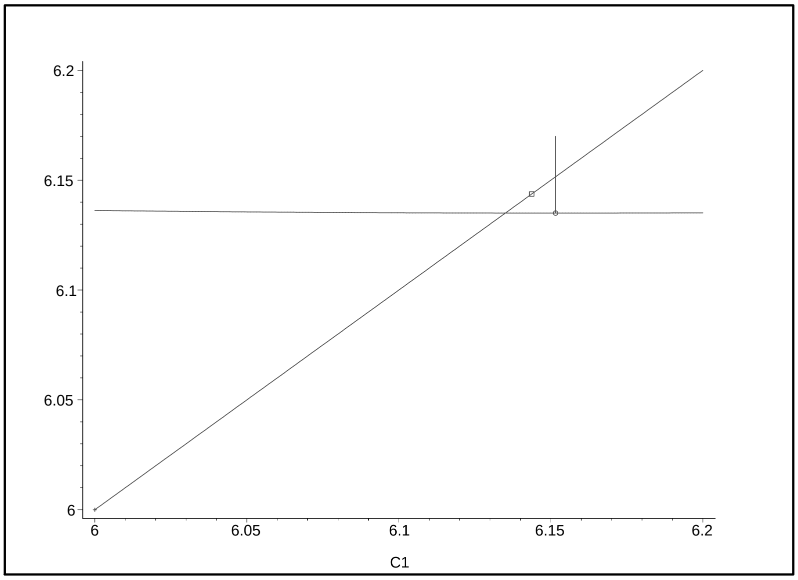

Figure 6: C^*_{eq} (vertical line), $\phi_{2,eq}(C)$ (curved line), D^*_{eq} (circle), C^*_{diag} (square), $(6,6)$ (cross) and the diagonal, $p = p' = 0.09$

We have $w_{C^*_{eq},D^*_{eq}} = 0.535056305018\ldots$, this the maximum, but we can not use it. $w_{C^*_{eq},C^*_{eq}} = 0.535055655126\ldots$, $w_{C^*_{diag},D^*_{diag}} = 0.535055963810\ldots$ is the optimal diagonal continuous value. $w_{6,6} = 0.534951097574\ldots$ is the optimal useful discrete value. We observe the order: $w_{C^*_{eq},D^*_{eq}} > w^*_{C^*_{diag},D_{diag}} > w_{C^*_{eq},C^*_{eq}} > w_{6,6}$.

We notice that, even if $p > p'$, we can have a similar situation. If we choose for instance $p = 0.09, p' = 0.08999$, we have the case described in Figure 7 and, with a closer look, in Figure 8, where the discrete optimal point $(6,6)$ is on the diagonal. This confirms to the existence of $\gamma_3(p)$ defined above.

A plot of $w_{C,D}, C \leq D$ and $w_{D,C}, C \geq D$,$p = p' = 0.09$ is given in Figure 9. This surface is symmetric w.r.t. the diagonal.

3 The x-strategy

We recall the notion of an x-strategy given in the Introduction: let $U_i, i = 1, 2, \ldots, n$ be independent random variables uniformly distributed on the interval $[0,1]$. Let $T_i = U_{\{i\}}$: T_i is the ith order statistic of the U_i's. T_i is the arrival time of X_i. The strategy is to wait until some time x^*_n and from x^*_n on, we select the first $X_i = +1$ or $X_i = -1$, using the previous algorithm with $p = p'$. Following Bruss [3], we call this strategy an x-strategy. In [10], the author gives, for this problem, the optimal x^*_n and the corresponding success probability P^*_n. In this Section, we analyze accordingly asymptotic expansions for $p = p'$. We also consider the success probability for small p, and also the case $p > p'$.

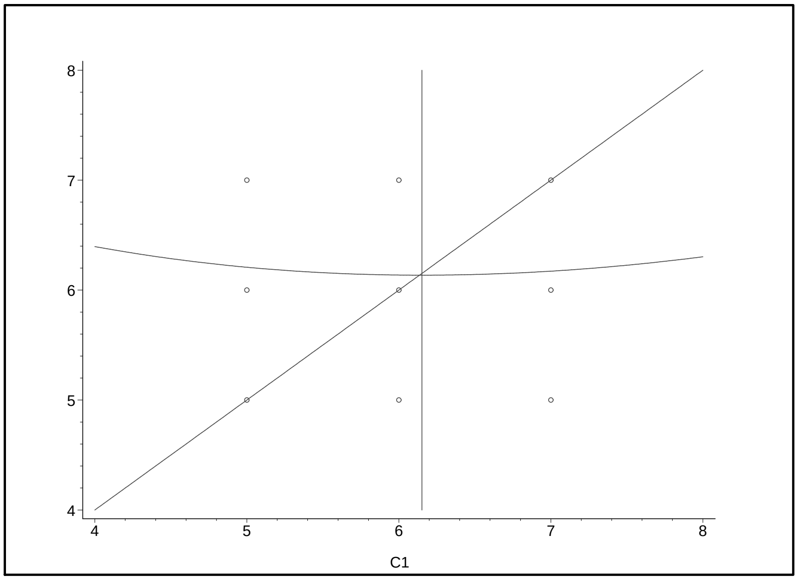

Figure 7: C^* (vertical line), $\phi_2(C)$ (curved line) , $p = 0.09, p' = 0.08999$, and some closest discrete points

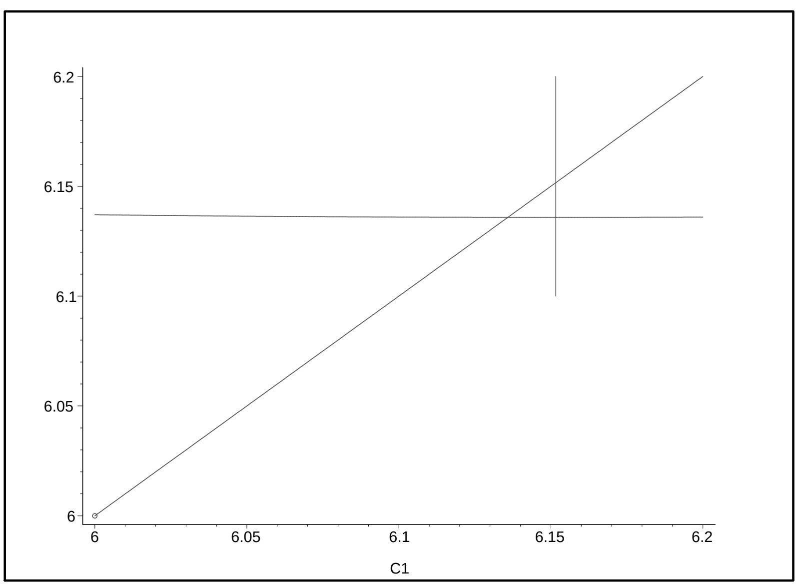

Figure 8: Closer look at Fig. 7, with optimal point $(6, 6)$

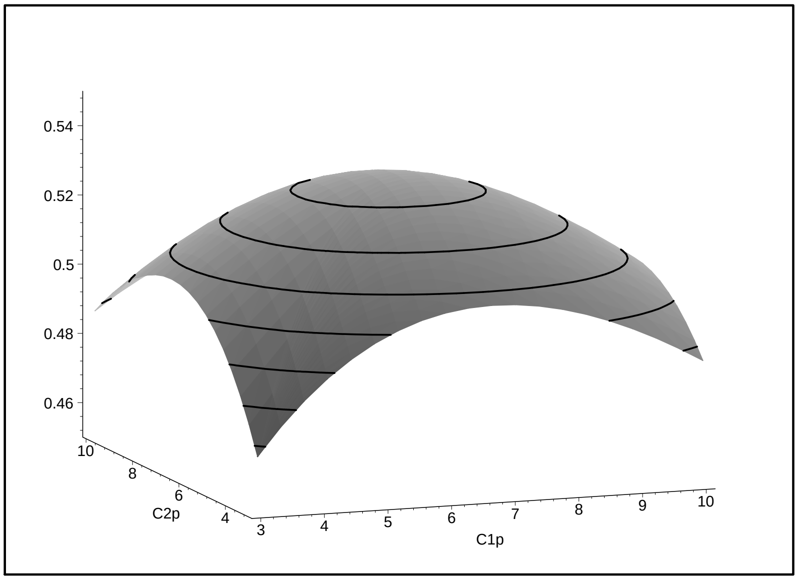

Figure 9: $w_{C,D}, C \leq D$ and $w_{D,C}, C \geq D$,$p = p' = 0.09$

3.1 The x-strategy, $p = p'$

Let first recall a few results from [10]. If we denote by ℓ the number of observed variables, starting from x, we must set, in (9), $C = \ell - 1$. This leads to the success probability

$$P_n(x,p) = \sum_0^n \binom{n}{\ell} (1-x)^\ell x^{n-\ell} 2 \left(q^\ell - \tilde{q}^\ell \right) = 2\,\left((q + p\,x)^n - \,(2\,q - 1 + 2\,p\,x)^n \right).$$

The optimal value x_n^* is solution of $\frac{dP_n(x,p)}{dx} = 0$, which leads to

$$x_n^* := \frac{-q + 2\,\beta_n\,q - be}{1 - q - 2\,\beta_n + 2\,be\,q}, \quad \beta_n := 2^{1/(n-1)}.$$

This gives

$$P_n^* := P_n(x_n^*, p) := 2\,(2\,2^{(\frac{1}{n-1})} - 1)^{(1-n)}.$$

Notice that P_n^* *is independent of* p. Open Problem 1: why is it so? It appears that, for $p = \tilde{p}_n$, we have $x_n^* = 0$, with

$$\tilde{p}_n = \frac{\beta_n - 1}{2\beta_n - 1}.$$

We can also check that $P_n(0, \tilde{p}_n) = P_n^*$.

Let us now turn to the the asymptotic analysis of the case $p = p'$ and the corresponding behaviour for small p.

Asymptotically, we obtain, for $n \to \infty$,

$$x_n^* = 1 - \frac{\ln(2)}{np} + \frac{1}{2}\,\frac{\ln(2)\,(-2 + 3\ln(2))}{pn^2} + \mathcal{O}\left(\frac{1}{n^3}\right), \tag{10}$$

$$P_n^* = \frac{1}{2} + \frac{1}{2}\,\frac{\ln(2)^2}{n} + \frac{\frac{1}{4}\ln(2)^2\,(2 - 2\ln(2) + \ln(2)^2)}{n^2} + \mathcal{O}\left(\frac{1}{n^3}\right),$$

$$\tilde{p}_n = \frac{\ln(2)}{n} + \frac{-\frac{1}{2}\ln(2)\left(-2+3\ln(2)\right)}{n^2} + \mathcal{O}\left(\frac{1}{n^3}\right). \qquad (11)$$

P_n^* converges to 1/2 for $n \to \infty$ .For instance, $P_{500}^* = 0.500480981417\ldots$. An interesting question is: what is the behaviour of P_n^* for $p \le \tilde{p}_n$? Following (11), we tentatively set $q = 1 - y/n, x = 0$ in $P_n(x,p)$. This leads to

$$P_n(y) = 2\,e^{-y} - 2\,e^{-2y} + \frac{-e^{-y}\,y^2 + 4\,e^{-2y}\,y^2}{n} + \frac{2\,e^{-y}\,(-\frac{1}{3}\,y^3 + \frac{1}{8}\,y^4) - 2\,e^{-2y}\left(-\frac{8}{3}\,y^3 + 2\,y^4\right)}{n^2} + \mathcal{O}\left(\frac{1}{n^3}\right).$$

In order to check, we put the first term of $\tilde{p}_n$ i.e. $y = \ln(2)$ into $P_n(y)$. Expanding, this leads to the first two terms of P_n^*. Similarly, putting the first two terms of $\tilde{p}_n$, i.e. $y = \ln(2) + \frac{-\frac{1}{2}\ln(2)\left(-2+3\ln(2)\right)}{n}$ into $P_n(y)$ gives the first three terms of P_n^*.

3.2 The x-strategy for $p > p'$

This case was not considered before. We can still use the x-strategy, but now we must set $D = \ell - 1$. Also, if $D \ge C_d^*$, we use $w_{C_d^*,D}$ and if $D \le C_d^*$, we use $w_{D,D}$ (we must stay above the diagonal). This leads to

$$\begin{aligned} P_n^* &= \sum_{\ell=C_d^*}^{n} \binom{n}{\ell}(1-x)^\ell x^{n-\ell} w_{C_d^*,\ell-1} + \sum_{l=0}^{C_d^*} \binom{n}{\ell}(1-x)^\ell x^{n-\ell} w_{\ell-1,\ell-1} \\ &= \sum_{0}^{n} \binom{n}{\ell}(1-x)^\ell x^{n-\ell} w_{C_d^*,\ell-1} + \sum_{l=0}^{C_d^*} \binom{n}{\ell}(1-x)^\ell x^{n-\ell}[w_{\ell-1,\ell-1} - w_{C_d^*,\ell-1}]. \end{aligned}$$

The first summation leads to $S_1 + S_2$, with

$$\begin{aligned} S_1 :=& \frac{\left(\frac{(1-x)\,q'}{x} + 1\right)^n x^n\,(-C_d^* - 1)\,p'}{q'} \\ &+ \frac{\left(\frac{(1-x)\,q'}{x} + 1\right)^n (1-x)\,q'\,n\left(-\frac{x^n\,C_d^*\,p'}{q'} - \frac{x^n\,(-C_d^* - 1)\,p'}{q'}\right)}{x\left(\frac{(1-x)\,(1-p')}{x} + 1\right)}, \end{aligned}$$

$$S_2 := \left(\frac{(1-x)\,q'}{x} + 1\right)^n x^n\,q'^{-C_d^*-1}\left(\frac{p\,(q^{C_d^*+1} - \tilde{q}^{C_d^*+1})}{p'} + \frac{p'\,(q'^{C_d^*+1} - \tilde{q}^{C_d^*+1})}{p}\right).$$

The second summation leads to a complicated expression, involving binomials and hypergeometric terms that we do not display here. However, if we plug in numerical values, for instance $p = 0.09, p' = 0.05, n = 40, C_d^* = 7$, we obtain a tractable function $P(x)$ that we can differantiate, leading to $x^* = 0.667967251301\ldots$. This gives $P(x^*) = 0.523618813813\ldots$.

4 The x-strategy with incomplete information

Bruss suggested to analyze this strategy because incomplete information has an increased appeal for applications.

We will only consider the case $p = p'$. The other cases can similarly analyzed, with more complicated algebra. We will consider the cases p known, n unknown, then n known, p unknown and finally n, p unknown. Some simulations are also provided. In all our numerical expressions, we will use $n = 500, p = 0.03$.

4.1 The case p known, n unknown

We will always denote by m the number of observed variables up to time x and by k the number of $\{+1, -1\}$ observed variables up to time x. From (10), we have $x_n^* \sim 1 - \frac{\ln(2)}{np}$ and we will use the natural estimate $\tilde{n} = \frac{m}{x}$. Hence we start from the formal equation resulting from (10), hence

$$x = 1 - \frac{x\ln(2)}{mp},$$

from which we deduce the two functions

$$x = g(m,p) = \frac{mp}{mp + \ln(2)},$$
$$m = f(x,p) = \frac{\ln(2)x}{p(1-x)}.$$

Our algorithm proceeds as follows: wait until m crosses the function $f(x,p)$ at value m^*. It follows from Bruss and Yor [9] ,Thm 5.1 that all optimal actions are confined to the interval $[x_1, 1]$ for some $x_1 < 1$ so that we can ignore preceding crossing, if any. (In the last-arrival problem, supposing no information at all, this value x_1 equals 1/2). The crossing algorithm gives a value $x^* = g(m^*, p)$. We will use this value in the x-strategy. First of all we notice that, asymptotically, m corresponds to a Brownian bridge of order $\sqrt{n}$ with a drift nx. On the other side, $f'(x_n^*) \sim pn^2/\ln(2)$. Hence, with high probability, m crosses $f(x,p)$ only once in the neighbourhood of x_n^*. Let

$$G(n,m,x) := \binom{n}{m} x^m (1-x)^{n-m}$$

be the distribution of m at time x. We have

$$\varphi(n,\mu,p) := \mathbb{P}(m^* = \mu) \sim G(n,\mu,g(\mu,p)),$$

and using

$$P_{eq}(\ell,p) := 2[q^\ell - \tilde{q}^\ell],$$

we obtain the success probability

$$P(n,p) = \sum_1^n \varphi(n,\mu,p) P_{eq}(n-\mu,p).$$

For instance, we show in Figure 10 an illustration of a typical crossing and in Figure 11, the function $\varphi(n,\mu,p)$ (line) together with $G(n,\mu,x_n^*)$ (circles) (the classical x-strategy μ distribution).

The distributions are quite similar. Open Problem 2: why? We obtain $P(n,p) \sim 0.5234\ldots$ (In the numerical summations, we sum μ from some value $\tilde{\mu}$ to avoid any problems near the origin)

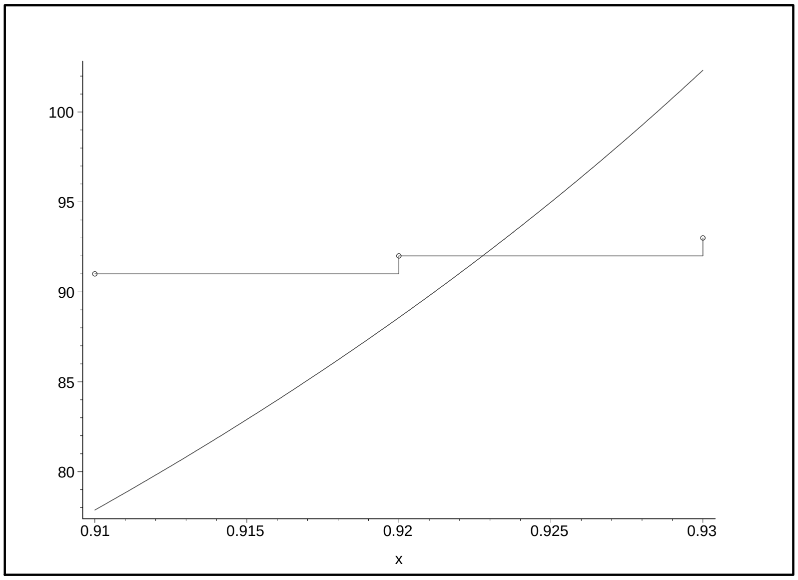

Figure 10: The case p known, n unknown: a typical crossing because it occurs close to 1

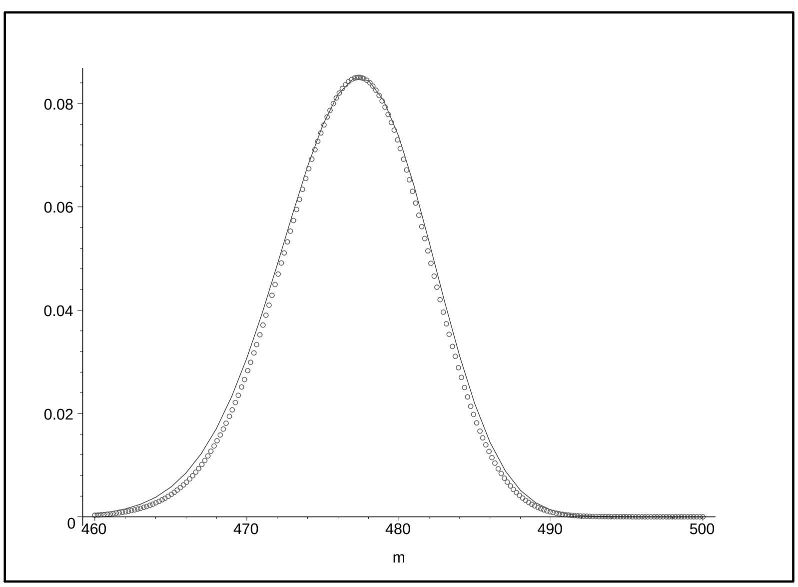

Figure 11: The case p known, n unknown: $\varphi(n, \mu, p)$ (line) , with $G(n, \mu, x_n^*)$ (circles)

4.2 The case n known, p unknown

Now we use the following estimate for $p : \tilde{p} = k/(2m)$. The formal starting equation is

$$x = 1 - \frac{\ln(2)}{np}.$$

Hence the two functions

$$x = u(n,p) = 1 - \frac{\ln(2)}{np},$$
$$p = h(n,x) = \frac{\ln(2)}{n(1-x)}.$$

The algorithm waits until $\tilde{p}$ crosses function $h(n,x)$ at value p^*, giving a value $x^* = u(n,p^*)$. Again, with high probability, $\tilde{p}$ crosses $h(n,x)$ only once in the neighbourhood of x_n^*. The joint distribution of m,k at time x is given, with $k \le m$ by

$$H(n,m,k,x,p) = G(n,m,x)\binom{m}{k}(2p)^k(1-2p)^{m-k}.$$

The joint distribution of $m = \mu, k$ *given* that $\tilde{p}$ has just crossed $h(n,x)$ is given by

$$\Pi(n,\mu,k,p) \sim H(n,\mu,k,u(n,\tilde{p}),p).$$

We have

$$\varphi(n,\mu,p) := \mathbb{P}(m^* = \mu) \sim \sum_{k=1}^{\mu} \Pi(n,\mu,k,p),$$

and finally the success probability is given by

$$P(n,p) = \sum_{1}^{n} \varphi(n,\mu,p) P_{eq}(n-\mu,p).$$

As an example, we show in Figure 12 the function $\varphi(n,\mu,p)$. Also $P(n,p) \sim 0.4927\ldots$

4.3 The case n,p unknown

The estimates are now $\tilde{p} = k/(2m), \tilde{n} = \frac{m}{x}$. This leads to formal starting equation

$$x = 1 - \frac{2x\ln(2)}{k}.$$

Hence the two functions

$$x = v(k) = \frac{k}{k + 2\ln(2)},$$
$$k = w(x) = \frac{2\ln(2)x}{(1-x)}.$$

The algorithm waits until k crosses function $w(x)$ at value k^*, giving a value $x^* = v(k^*)$. Again, with high probability, k crosses $w(x)$ only once in the neighbourhood of x_n^*. The joint distribution of $m = \mu, k$ *given* that k has just crossed $w(x)$ is given by

$$\Pi(n,\mu,k,p) \sim H(n,\mu,k,v(k),p).$$

We have

$$\varphi(n,\mu,p) := \mathbb{P}(m^* = \mu) \sim \sum_{k=1}^{\mu} \Pi(n,\mu,k,p),$$

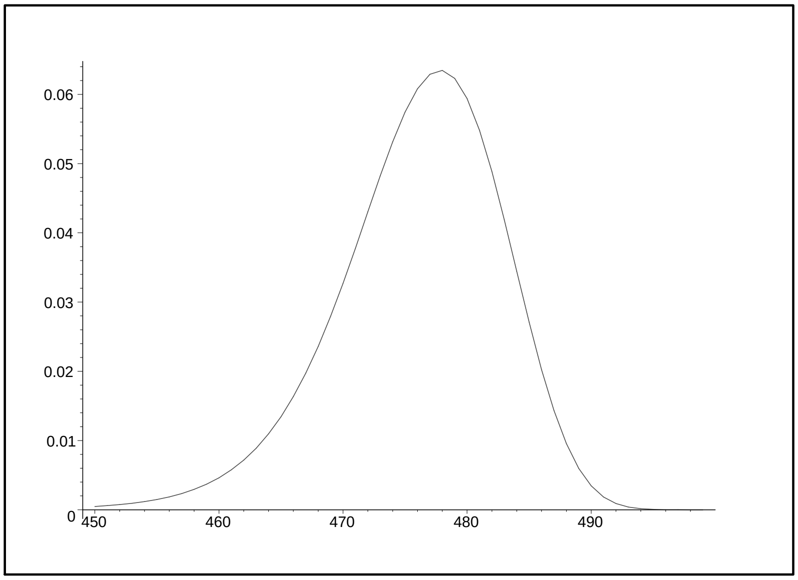

Figure 12: The case n known, p unknown: $\varphi(n, \mu, p)$

and finally the success probability is given by

$$P(n,p) = \sum_{1}^{n} \varphi(n, \mu, p) P_{eq}(n - \mu, p).$$

For instance, we show in Figure 13 the function $\varphi(n, \mu, p)$ together with the corresponding distribution in the the case n known, p unknown (circles). Curiously enough, the distributions are quite similar but different from the case p known, n unknown. Open Problem 3: why? Also $P(n,p) \sim 0.5156\ldots$.

4.4 Simulations

We have made three simulations of the crossing value μ distribution compared with $\varphi(n, \mu, p)$. Each time we made 500 simulated paths. For the case p known, n unknown, a typical path is given in Figure 14 and, in Figure 15 , we show the empirical observed distribution, together with $\varphi(n, \mu, p)$ (For the purpose of smoothing, we have grouped two successive observed probabilities together). Numerically, this gives $P_{sim}(n,p) = 0.4981\ldots$.

Similarly, for the case n known, p unknown, a typical path is given in Figure 16 and, in Figure 17 , we show the empirical observed distribution, together with $\varphi(n, \mu, p)$. Numerically, this gives $P_{sim}(n,p) = 0.4915\ldots$.

For the case n, p unknown, a typical path is given in Figure 18 and, in Figure 19 , we show the empirical observed distribution, together with $\varphi(n, \mu, p)$. Numerically, this gives $P_{sim}(n,p) = 0.4805\ldots$.

All fits are satisfactory.

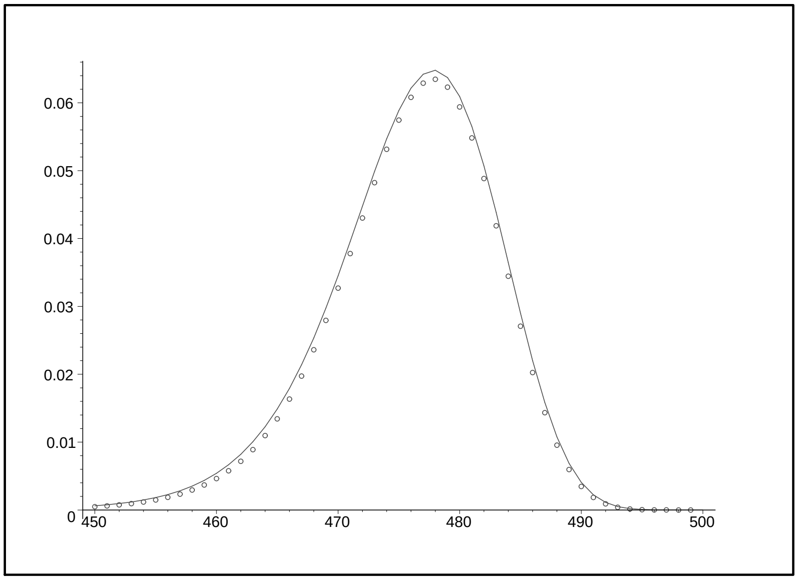

Figure 13: The case n, p unknown: $\varphi(n, \mu, p)$ (line) together with the corresponding distribution in the the case n known, p unknown (circles)

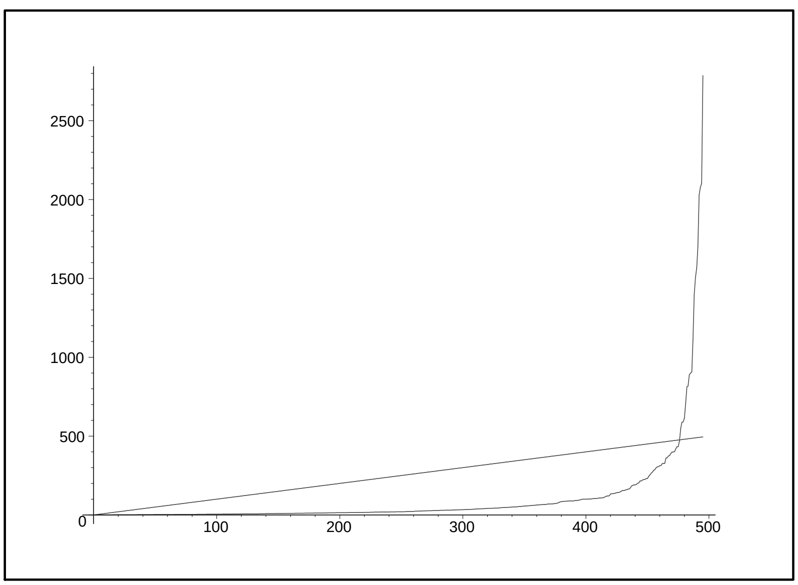

Figure 14: The case p known, n unknown: a typical path

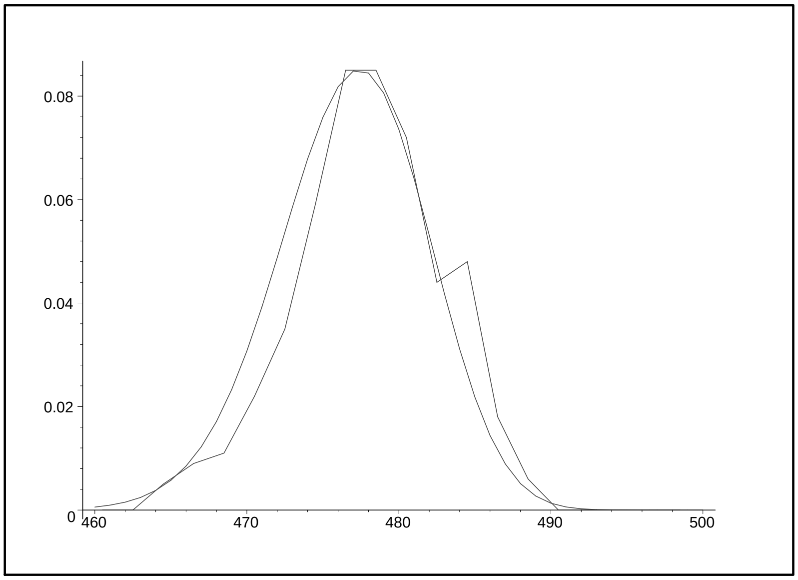

Figure 15: The case p known, n unknown: the empirical observed distribution, together with $\varphi(n, \mu, p)$

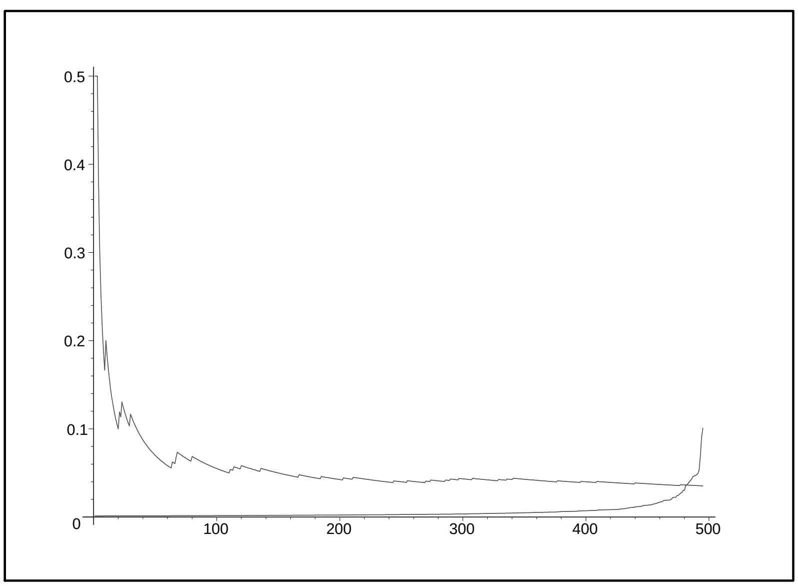

Figure 16: The case n known, p unknown: a typical path

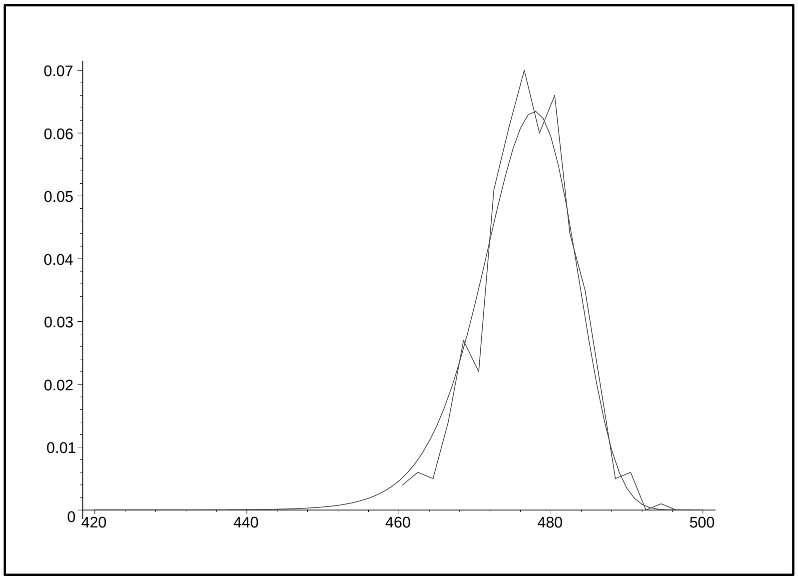

Figure 17: The case n known, p unknown: the empirical observed distribution, together with $\varphi(n, \mu, p)$

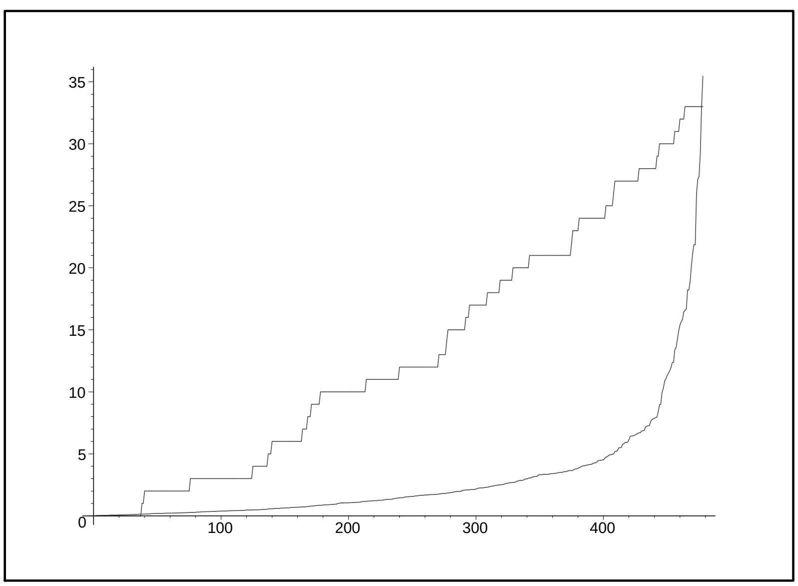

Figure 18: The case n, p unknown: a typical path

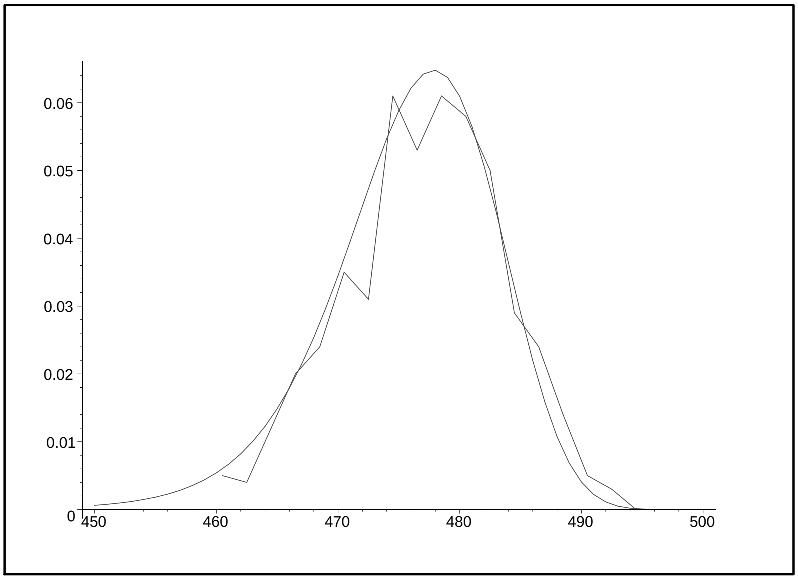

Figure 19: The case n, p unknown: the empirical observed distribution, together with $\varphi(n, \mu, p)$

5 Conclusion

Using a continuous model, some asymptotic expansions and an incomplete information strategy, we have obtained a refined and asymptotic analysis of the extended Weber problem and several versions of Bruss-Weber problems. Three problems remain open: why is P_n^* independent of p? Can we justify the similarities in the distributions of the crossing value m^*? An interesting problem would be to consider the case with several values $\{-k, -(k-1), \ldots, -1, 0, 1, \ldots, k\}$ with corresponding stopping times. If moreover values can be associated with relative ranks, such problems (Bruss calls them “ basket ” problems) are partially studied in Dendievel [11].

6 Appendix. An asymptotic analysis of $\gamma_4(p)$

Some numerical experiments show that, for C^* near -1, $\gamma_4(p)$ is very close to $p' = 1 - p$, and that no value $C^* < -1$ appears as solution of (5). The asymptotic behaviour of $\gamma_4(p)$ for C^* near -1 can be summarized as follows. We keep only dominant terms in our expansions.

- for p near 1, we set $p' = w$. For $w = 0$, $\phi_1(C)$ is identically 0. So we expand (5) near $w = 0$ and keep the w term. This gives

  $$p^2(1-p)^{C^*}(1 + C^* \ln(1-p) + \ln(1-p)) = 0.$$

  Setting $p = 1 - \xi, C^* = -1 + \eta$, we obtain

  $$(-1+\xi)^2 \xi^{-1+\eta}(1 + \eta \ln(\xi)) = 0,$$

  hence

  $$\eta(\xi) \sim -1/\ln(\xi), \xi \to 0,$$
  $$\xi(\eta) \sim \exp(-1/\eta), \eta \to 0.$$

  For instance, for $C^* = -1 + 0.09$ we have ($\hat{x}$ always denotes some solution of (5))
  $\hat{\xi} = 0.00001494533852483\ldots$ and $\eta(\hat{\xi}) = 0.09000000000002\ldots, \xi(0.09) = 0.00001494533852478\ldots.$

- on the diagonal $p' = p$, we set $p = p' = 1/2 - \varepsilon, C^* = -1 + \eta$. From (8), expand w.r.t. ξ, we obtain

$$C^* \sim \frac{\ln(-16\ln(2) - 8\ln(\varepsilon)) + \ln(\varepsilon)}{-2\ln(2) - \ln(\varepsilon)} \sim -1 - \frac{\ln(2) + \ln(-\ln(\varepsilon))}{\ln(\varepsilon)},$$

hence

$$\eta(\xi) \sim -\frac{\ln(2) + \ln(-\ln(\varepsilon))}{\ln(\varepsilon)}, \varepsilon \to 0.$$

To obtain ε as a function of η, we set $A := -\ln(\varepsilon)$. We derive, to first order,

$$\begin{aligned}
&\ln(2) + \ln(A) - \eta A = 0, \\
&A\exp(-\eta A) = 1/2, \\
&-\eta A\exp(-\eta A) = -\eta/2, \\
&-\eta A = W_{-1}(-\eta/2), \\
&A = -W_{-1}(-\eta/2)/\eta, \text{ for } -\eta/2 > -1/e = -0.3678794411\ldots, \\
&\varepsilon(\eta) \sim \exp(W_{-1}(-\eta/2)/\eta), \eta \to 0,
\end{aligned}$$

where $W(x)$ is the Lambert-W function and the lower branch has $W \leq -1$ and is denoted by $W_{-1}(x)$. It decreases from $W_{-1}(-1/e) = -1$ to $W_{-1}(0) = -\infty$. For instance, for $\varepsilon = 10^{-20}, \hat{\eta} = 0.1005777569\ldots$ and $\eta(\hat{\varepsilon}) = .09821378400137\ldots, \varepsilon(\hat{\eta}) = 4.10^{-20}$.

Now $W_{-1}(x) \sim \ln(-x), x \uparrow 0$. Hence

$$\varepsilon(\eta) \sim \exp(\ln(\eta/2)/\eta), \eta \to 0.$$

- in the neighbourhood of $p' = 1 - p$, we set $p' = 1 - p - \delta, C^* = -1 + \eta$. Hence $\tilde{q} = \delta, q' = p - \delta$. As $\delta \to 0$, we have $p' \sim 1 - p, q' \sim p$. So we expand (5) to first order. We obtain

$$C_1\delta^\eta + C_2 q^\eta + C_3(p - \delta)^{-1+\eta} = 0,$$

with

$C_1 = C_4 + C_5\ln(\delta), C_4 = (p^2 + (1-p)^2)\ln(p), C_5 = -p^2 - (1-p)^2, C_2 = -p^2(-\ln(q) + \ln(p)), C_3 = -(1-p)^2 p.$

This leads to

$$\eta(\delta) \sim \frac{\ln(C_6) - \ln(\ln(\delta))}{\ln(\delta)}, \delta \to 0,$$

$$C_6 = \frac{-1 + \ln(1/pq)p^3 - 2\ln(1/pq)p^4 + \ln(1/pq)p^5}{p(2p^2 + 1 - 2p)(p-1)^2}.$$

Setting $B := -\ln(\delta), C_7 = -C_6$, this leads to

$$\begin{aligned}
&Be^{-\eta B} \sim C_7, \\
&-\eta Be^{-\eta B} \sim -\eta C_7, \\
&-\eta B \sim W_{-1}(-\eta C_7), \\
&B \sim -W_{-1}(-\eta C_7)/\eta, \\
&\delta(\eta) \sim \exp(\ln(\eta C_7)/\eta), \eta \to 0, \\
&\eta(\delta) \sim (\ln(B) - \ln(C_7))/B, \delta \to 0.
\end{aligned}$$

For instance, for $p = 0.75, \eta = 0.035$, we obtain $\hat{p'} = 0.25 - 0.161018555971\ldots 10^{-60}, \hat{\delta} = .161018555971\ldots 10^{-60}, C_6 = -35.12208439\ldots$ and $\eta(\hat{\delta}) = .009877595163\ldots$. $-\ln(\delta) = 139\ldots$ is not large enough, compared with $C_7 = 34\ldots$ in order to use $\eta(\hat{\delta})$. However, $\ln(B)/B = .03530119866$ which is quite satisfactory. On this other side, $\eta C_7 = 1.22\ldots$, which is too large ($> 1/e$) in our case for allowing using $-W_{-1}(-\eta C_7)/\eta$.

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